Indian J. pure appl. Math., **12**(10) : 1187–1191, October 1981

ON CHARACTERIZING GRAPHS SWITCHING EQUIVALENT TO ACYCLIC GRAPHS

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(*Received* 20 *February* 1981; *after revision* 25 *May* 1981)

We consider the problem of characterizing graphs which can be switched to an acyclic graph. We give first an existential characterization of such graphs in terms of certain partitions of the point set of the given graph into two subsets, and then deduce that the graphs cannot contain K_5 and C_n, $n \geqslant 7$, as induced subgraphs.

1. SWITCHING EQUIVALENCE

For standard terminology and notation in graph theory we refer the reader to the text-book by Harary (1972).

A marking μ of a graph $G = (V, E)$ is a function $\mu : V \to \{+, -\}$ which assigns a sign (or colour) + (plus, or black) or − (minus, or white) to each vertex of G. We write G_μ to indicate that G is a graph together with a marking μ of its vertices—we refer to G_μ as a marked graph (cf. Beineke and Harary 1978, Harary *et al.* 1977). We let $M(G)$ denote the set of all markings of G. For any $\mu \in M(G)$, points in G_μ which are marked + (respectively, −) are said to be positive (respectively, negative) — further, G_μ is said to be all-positive (respectively, all-negative) if $\mu(u) = +$ (−, respectively) for all $u \in V(G)$, and we shall say that G_μ is monochromatic if it is either all-positive or all-negative.

Switching a graph G with respect to a marking $\mu \in M(G)$ means deleting all lines whose ends are of opposite signs in G_μ and introducing a new line between two points of opposite signs whenever they were non-adjacent in G_μ (cf. Van-Lint and Seidel 1966). The graph obtained from G_μ after switching, denoted $\mathcal{S}(G_\mu)$, is called the switched graph. These notions are illustrated by means of graphs shown in Fig. 1.

We say that a graph G switches to a graph H if there exists $\mu \in M(G)$ such that $\mathcal{S}(G_\mu) = H$ where the symbol " = " between two graphs means the usual isomorphism between them. We write $G \sim H$ if G switches to H. The relation ' $\sim$ ' so defined is an equivalence relation on the set $\mathcal{G}$ of all graphs and we can speak about switching classes of graphs. Any two graphs in a switching class are said to be switching

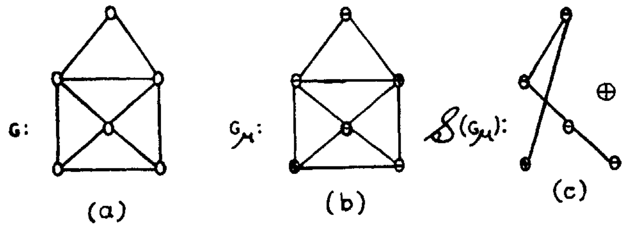

FIG. 1. Switching a graph with respect to one of its markings.

equivalent to each other. Thus, for example, the graph of Fig. 1 (a) switches to $P_5 \cup K_1$, and hence is switching equivalent to it.

In this note, we consider the problem of characterizing graphs which are switching equivalent to acyclic graphs. Such a characterization is found to be useful in many theoretical contexts, especially in the problems involving graph equations with respect to switching equivalence (e.g., see Acharya 1980; Cvetković and Simić 1978, 1980).

2. GRAPHS SWITCHING EQUIVALENT TO ACYCLIC GRAPHS

We first give an existential characterization of graphs which switch to acyclic graphs.

Theorem 2.1 — A graph G switches to an acyclic graph if and only if there exists a partition P of $V(G)$ into two subsets V_1 and V_2 such that the following three conditions are satisfied:

C1 : $\langle V_i \rangle_G$, the subgraph of G induced by V_i, is acyclic, $i = 1, 2$.

C2 : If $H_1, \ldots, H_m$ are the components of $\langle V_1 \rangle_G$, and $L_1, \ldots, L_n$ are the components of $\langle V_2 \rangle_G$, then for any i, j, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, there is at most one pair of non-adjacent points u, v with $u \in V(H_i)$ and $v \in V(L_j)$.

C3 : The bipartite graph $B_P(G)$, whose points are $H_1, \ldots, H_m$ and $L_1, \ldots, L_n$, and whose lines are precisely those pairs H_iL_j for which there is exactly one pair of non-adjacent points, u, v with $u \in V(H_i)$ and $v \in V(L_j)$, is acyclic.

PROOF : *Sufficiency* — Suppose that G has a partition $P = \{V_1, V_2\}$ of the type mentioned in the theorem. Consider $\mu \in M(G)$ defined by:

$$\mu(u) = \begin{cases} + \text{ if } u \in V_1 \\ - \text{ if } u \in V_2. \end{cases} \qquad \ldots(2.1)$$

We claim that $\mathcal{S}(G_\mu)$ is acyclic. Suppose that $\mathcal{S}(G_\mu)$ contains a cycle C_0. Then C1 implies that C_0 contains an even number, say $2r (r \geqslant 1)$, of lines each of which has an end in V_1 and the other in V_2. According to C2, no two of these lines join points of the same pair of components $H_i \in \langle V_1 \rangle_G$, $L_j \in \langle V_2 \rangle_G$. Hence, each of these lines $x = uv$ in $\mathcal{S}(G_\mu)$ defines a unique line $X = H_i L_j$ in $B_P(G)$ where H_i and L_j are the components of $\langle V_1 \rangle_G$ and $\langle V_2 \rangle_G$, respectively, for which $u \in V(H_i)$ and $v \in V(L_j)$. Furthermore, these lines X in $B_P(G)$ constitute a cycle of length $2r$. This contradicts C3. Thus, it follows that $\mathcal{S}(G_\mu)$ is acyclic.

Necessity — To prove the converse, suppose that G switches to an acyclic graph via a marking $\mu \in M(G)$. Let

$$V_1 = V^+(G) = \{u \in V(G_\mu)/\mu(u) = +\}$$

$$V_2 = V^-(G) = \{u \in V(G_\mu)/\mu(u) = -\}.$$

Let $H_1, \ldots, H_m$ and $L_1, \ldots, L_n$ be the components of $\langle V_1 \rangle_G$ and $\langle V_2 \rangle_G$, respectively. Since $\mathcal{S}(G_\mu)$ is acyclic G_μ cannot contain any monochromatic cycles, and hence C1 follows.

Next, if C2 is not true, then for some H_i and L_j there exist at least two pairs of non-adjacent points u, v and x, y such that $u, x \in V(H_i)$ and $v, y \in V(L_j)$ (so, at least three of these points are distinct). Then in $\mathcal{S}(G_\mu)$ we must have lines uv and xy. But then since H_i and L_j remain intact in $\mathcal{S}(G_\mu)$ it follows that u, v, x and y are contained in a cycle in $\mathcal{S}(G_\mu)$, a contradiction. Thus, C2 must be true.

Lastly, let $P = \{V_1, V_2\}$ and, suppose that $B_P(G)$ contains a cycle

$$C = (H_{i_1}, L_{j_1}, H_{i_2}, L_{j_2}, \ldots, L_{j_k}, H_{i_1}).$$

(This cycle is of even length $2k$ as $B_P(G)$ is bipartite.) Then, by C2, each line $H_{i_r} L_{j_s}$ of this cycle represents exactly one pair of non-adjacent points u, v in G_μ with $u \in V(H_{i_r})$ and $v \in V(L_{j_s})$, and each such pair forms a line in $\mathcal{S}(G_\mu)$. Let $u_r x_r$ be the line of $\mathcal{S}(G_\mu)$ corresponding to the line $H_{i_r} L_{j_r}$ of the cycle C with $u_r \in V(H_{i_r})$, $x_r \in V(L_{j_r})$, $r = 1, \ldots, k$, $y_r v_{r+1}$ be the line of $\mathcal{S}(G_\mu)$ corresponding to the line $L_{j_r} H_{i_{r+1}}$ of C with $y_r \in V(L_{j_r})$ and $v_{r+1} \in H_{i_{r+1}}$, $r = 1, \ldots, k-1$, and $y_k v_1$ be the line of $\mathcal{S}(G_\mu)$ corresponding to the line $L_{j_k} H_{i_1}$ of C with $y_k \in V(L_{j_k})$ and $v_1 \in V(H_{i_1})$. Here, for any $r = 1, \ldots k$, u_r and v_r or x_r and y_r may be equal or distinct—furthermore, whenever they are distinct they are joined by a (unique) path within H_{i_r} and L_{j_r}, respectively, in $\mathcal{S}(G_\mu)$. Writing $u_r - v_r$ and $x_r - y_r$ for such a path within H_{i_r} and L_{j_r} respectively (when $u_r = v_r$, or $x_r = y_r$, we have $u_r - v_r = v_r$, or $x_r - y_r = y_r$, the path of length zero) we can then trace a cycle in $\mathcal{S}(G_\mu)$ by traversing the paths $u_1 x_1$, $x_1 - y_1$, $y_1 u_2$, $u_2 - v_2$, $v_2 x_2$, $x_2 - y_2$, $\ldots$, $u_k - v_k$, $v_k x_k$, $x_k - y_k$, $y_k v_1$, $v_1 - u_1$ in that order. This is a contradiction to the hypothesis, and hence C3 must be true.

This completes the proof. ||

Theorem 1.2 — If a graph switches to an acyclic graph then it cannot contain an induced subgraph isomorphic to either K_5 or C_n, $n \geqslant 7$.

PROOF : It is enough to show that K_5 and C_n, $n \geqslant 7$, do not switch to an acyclic graph. As for K_5, in any partition of the points of K_5 into two sets V_1 and V_2 we would find at least three points which are mutually adjacent in one of V_1 and V_2. Then by Theorem 1.1(C1) it follows that K_5 does not switch to an acyclic graph.

We shall next show that C_n, $n \geqslant 7$, cannot be switched to an acyclic graph. Towards this end, suppose that there exists $\mu \in M(C_n)$, $n \geqslant 7$, such that $\mathcal{S}((C_n)_\mu)$ is acyclic. Then, by Theorem 1.1, the partition $P = \{V^+, V^-\}$ of $V((C_n)_\mu)$ satisfies the conditions C1 — C3.

We first claim that no component of either $\langle V^+ \rangle$ or $\langle V^- \rangle$ is nontrivial. Towards this end, suppose that one of $\langle V^+ \rangle$ and $\langle V^- \rangle$ has a nontrivial component. Without loss of generality, let $\langle V^- \rangle$ have a nontrivial component L. Clearly, L must be a path $(u_0, u_1, \ldots, u_k)$ with $k \geqslant 1$. Hence u_0 and u_k are joined to some points u, v of V^+, respectively. If $k \geqslant 2$ then $u = v$ for otherwise since the degree of every point of C_n is two we find that u is non-adjacent to u_{k-1} and u_k, a contradiction to C2. Now, if $u = v$ then we get $\langle V^+ \rangle = K_1$ and $\langle V^- \rangle = L$ so that $n = k + 2$, and since $n \geqslant 7$ we get $k \geqslant 5$. But then since C_n is 2-regular it follows that u is non-adjacent to each of $u_1, \ldots, u_{k-1}$, a contradiction to C2. Therefore, we may assume $L = K_2$. Further, u and v cannot be adjacent as $n \geqslant 7$, and hence there exists a point $w \in V(C_n) - \{u, u_0, u_1\}$ such that $vw \in E(C_n)$. If $w \in V^+$ then w is non-adjacent to u_0 and u_1, a contradiction to C2. Therefore, $w \in V^-$. Then w belongs to a component L', different from L, of $\langle V^- \rangle$. Now, $uw \notin E(C_n)$ as $n \geqslant 7$ and since C_n is 2-regular we must have a point x, different from u, v, adjacent to w in C_n. Then $x \notin V^-$ for if $x \in V^-$ we have $ux \notin E(C_n)$ as $n \geqslant 7$ and hence u is non-adjacent to both w and x, a contradiction to C2. Then, $x \in V^+$ whence x is non-adjacent to both the points u_0 and u_1, a contradiction to C2. Thus, we conclude that no component of either $\langle V^+ \rangle$ or $\langle V^- \rangle$ is nontrivial as claimed. This implies that n is even so that $n = 2r \geqslant 8$ and $|V^+| = |V^-| = r \geqslant 4$.

Now, let $V^+ = \{a_1, \ldots, a_r\}$ and $V^- = \{b_1, \ldots, b_r\}$. If $r = 4$ then $(a_1, b_2, a_4, b_1, a_3, b_4, a_2, b_3, a_1) = \mathcal{S}((C_8)_\mu)$, a contradiction to the supposition that μ switches C_8 to an acyclic graph. Next, if $r \geqslant 5$, we can find the 4-cycle (i.e., cycle of length 4) $(a_1, b_3, a_2, b_4, a_1)$ in $\mathcal{S}((C_n)_\mu)$, again a contradiction to our supposition that μ switches C_n to an acyclic graph.

The foregoing arguments establish that C_n, $n \geqslant 7$, cannot be switched to an acyclic graph, and hence the proposition follows. ||

Theorem 1.2 indicates that the graphs which switch to acyclic graphs can be characterized by means of forbidding some graphs as induced subgraphs—such characterizations are known in graph theory as "forbidden subgraph characterizations". Thus, we have shown above that K_5 and C_n, $n \geqslant 7$, are forbidden in a graph switching equivalent to an acyclic graph. To find a complete list of forbidden subgraphs for graphs switching equivalent to acyclic graphs is an open problem which we shall treat separately elsewhere.

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