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AN ORDER RESULT INVOLVING THE σ-FUNCTION

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In this paper we establish an asymptotic formula for the sum $\sum_{\substack{m \leq x \\ t \mid m}} \sigma^r(m)$ where $\sigma(m)$ denotes the sum of the divisors of m and r, t are fixed positive integers. Using a result of Walfisz (1963), we obtain an improvement in the O-estimate of the error term established by Mirsky (1949).

1. INTRODUCTION

Throughout the paper, m, n, t denote positive integers and x denotes a real variable $\geqslant 2$. The letter p is reserved for primes. Let $\tau(m)$ be the number of divisors of m, $\sigma(m)$ be the sum of the divisors of m and $\varphi(m)$ be the Euler-totient function.

Asymptotic formulae for the sums

$$\sum_{m \leq x} \tau^r(m) \quad \text{and} \quad \sum_{m \leqslant x} \varphi^r(m),$$

where r is any positive integer, have been established by Ramanujan [1916, eqn. (8)] and Chowla (1929-30) respectively. The object of this paper is to establish an asymptotic formula for the sum $\sum_{m \leqslant x} \sigma^r(m)$, for $r \geqslant 2$. In the particular case, when $r = 2$, the asymptotic formula was stated by Ramanujan [1916, eqn. (19)]. In fact, we establish (see Theorem in section 3) an asymptotic formula for the more general sum $\sum_{\substack{m \leq x \\ t \mid m}} \sigma^r(m)$.

2. PRELIMINARIES

Let μ be the Möbius function and $\theta(m)$ be the number of square-free divisors of m. We need the following best-known estimate due to Walfisz (1963) concerning the average of the function $\sigma(m)$:

Lemma 2.1 (cf. Walfisz 1963, Satz 4, p. 99) — For $x \geqslant 2$,

$$\sum_{m \leqslant x} \sigma(m) = \frac{\pi^2 x^2}{12} + O(x \log^{2/3} x).$$

Lemma 2.2 — Let $\sigma(m; n)$ denote the sum of the divisors of m which are prime to n. Then we have

$$\sum_{m \leqslant x} \sigma(m; n) = \frac{\pi^2 x^2 \varphi(n)}{12n} + O(\theta(n)\, x \log^{2/3} x)$$

where the O-estimate is uniform in x and n.

PROOF : By Lemma 2.1, we have

$$\sum_{m \leqslant x} \sigma(m; n) = \sum_{\substack{d\delta \leqslant x \\ (d, n) = 1}} d = \sum_{d\delta \leqslant x} d \sum_{\substack{ab = d \\ a \mid n}} \mu(a)$$

$$= \sum_{a \mid n} a\mu(a) \sum_{ab\delta \leqslant x} b$$

$$= \sum_{a \mid n} a\mu(a) \sum_{k \leqslant x/a} \sigma(k)$$

$$= \sum_{a \mid n} a\mu(a) \left\{ \frac{\pi^2 x^2}{12a^2} + O(a^{-1} x \log^{2/3} x) \right\}$$

$$= \frac{\pi^2 x^2}{12} \sum_{a \mid n} \frac{\mu(a)}{a} + O\left(x \log^{2/3} x \sum_{a \mid n} \mu^2(a) \right)$$

$$= \frac{\pi^2 x^2 \varphi(n)}{12n} + O(\theta(n)\, x \log^{2/3} x).$$

Hence Lemma 2.2 follows.

Lemma 2.3 — For $x \geqslant 2$, we have

$$\sum_{\substack{m \leqslant x \\ t \mid m}} \sigma(m) = \frac{\pi^2 x^2 A_1(t)}{12t} + O(S(t)\, x \log^{2/3} x)$$

where the O-constant is independent of x and t,

$$A_1(m) = \sum_{q \mid m} \frac{\varphi(q)}{q^2} \qquad \text{...(2.1)}$$

and

$$S(m) \equiv \sum_{d \mid m} \frac{\mu^2(d)\, 2^{\omega(d)}}{\varphi(d)} = \prod_{p \mid m} \left(1 + \frac{2}{p-1}\right) \qquad \text{...(2.2)}$$

$\omega(m)$ being the number of distinct prime factors of m.

PROOF : By Lemma 2.2, we have

$$\sum_{\substack{m \leqslant x \\ t \mid m}} \sigma(m) = \sum_{\substack{d\delta \leqslant x \\ t \mid d\delta}} d = \sum_{\substack{d\delta \leqslant x \\ t(d,t)^{-1} \mid \delta}} d = \sum_{d(d,t)^{-1}k \leqslant x/t} d$$

$$= \sum_{q \mid t} \sum_{\substack{(d/q)\, k \leqslant x/t \\ (d,t)=q}} d = \sum_{q \mid t} q \sum_{\substack{\delta k \leqslant x/t \\ (\delta, t/q)=1}} \delta$$

$$= \sum_{q \mid t} q \sum_{m \leqslant x/t} \sum_{\substack{\delta k = m \\ (\delta, t/q)=1}} \delta = \sum_{q \mid t} q \sum_{m \leqslant x/t} \sigma(m; t/q)$$

$$= \sum_{q \mid t} q \left\{ \frac{\pi^2 x^2 \varphi(t/q)}{12 t^2 (t/q)} + O\left(\theta(t/q) t^{-1} x \log^{2/3} x\right) \right\}$$

$$= \frac{\pi^2 x^2}{12t} \sum_{q \mid t} \frac{\varphi(t/q)}{(t/q)^2} + O\left(x \log^{2/3} x \sum_{q \mid t} \frac{\theta(t/q)}{(t/q)} \right)$$

$$= \frac{\pi^2 x^2 A_1(t)}{12t} + O\left(x \log^{2/3} x S(t)\right),$$

since

$$\sum_{q \mid t} \frac{\theta(q)}{q} = \prod_{p^\alpha \| t} \left(1 + 2\left(\frac{1}{p} + \frac{1}{p^2} + \ldots + \frac{1}{p^\alpha}\right)\right)$$

$$\leqslant \prod_{p \mid t} \left(1 + 2\left(\frac{1}{p} + \frac{1}{p^2} + \ldots\right)\right)$$

$$= \prod_{p \mid t} \left(1 + \frac{2}{p-1}\right) = S(t),$$

by (2.2). Hence Lemma 2.3 follows.

Lemma 2.4 — For $x \geqslant 2$, we have

$$\sum_{\substack{m \leqslant x \\ t \mid m}} \sigma^2(m) = \frac{\pi^2 A_2(t)\, x^3}{18t} + O\left(x^2 \log^{5/3} x S(t)\right) \qquad \ldots(2.3)$$

where the O-constant is independent of x and t,

$$A_2(m) \equiv \sum_{k \mid m} \frac{B_1(k;\, m)}{k} = O\,(\tau(m) \log m) \qquad \ldots(2.4)$$

and for $k \mid m$,

$$B_1(k;\, m) = \sum_{\substack{n=1 \\ (n,\, m/k)=1}}^{\infty} \frac{A_1(nm)}{n^2} \qquad \ldots(2.5)$$

$A_1(m)$ and $S(m)$ being given by (2.1) and (2.2).

Proof : If $\{a, b\}$ denotes the least common multiple of a and b, then we have

$$\sum_{\substack{m \leqslant x \\ t \mid m}} \sigma^2(m) = \sum_{\substack{d\delta \leqslant x \\ t \mid d\delta}} d\sigma(d\delta) = \sum_{\delta \leqslant x} \frac{1}{\delta} \sum_{\substack{m \leqslant x \\ \delta \mid m \\ t \mid m}} m\sigma(m)$$

$$= \sum_{\delta \leqslant x} \frac{1}{\delta} \sum_{\substack{m \leqslant x \\ \{\delta,\, t\} \mid m}} m\sigma(m) \qquad \ldots(2.6)$$

Now, by Lemma 2.3 and partial summation, we have

$$\sum_{\substack{m \leqslant x \\ t \mid m}} m\sigma(m) = \frac{\pi^2 x^3 A_1(t)}{18t} + O\,(S(t)\, x^2 \log^{2/3} x). \qquad \ldots(2.7)$$

Making use of (2.7) in (2.6) we obtain

$$\sum_{\substack{m \leqslant x \\ t \mid m}} \sigma^2(m) = \sum_{\delta \leqslant x} \frac{1}{\delta} \left\{\frac{\pi^2 x^3 A_1(\{\delta,\, t\})}{18\{\delta,\, t\}} + O\,(S(\{\delta,\, t\})\, x^2 \log^{2/3} x)\right\}$$

$$= \frac{\pi^2 x^3}{18t} \sum_{\delta \leqslant x} \frac{A_1(\delta t/(\delta,\, t))\,(\delta,\, t)}{\delta^2}$$

$$+\, O\left(x^2 \log^{2/3} x \sum_{\delta \leqslant x} \delta^{-1} S(\delta t/(\delta,\, t))\right)$$

$$= \frac{\pi^2 x^3}{18t} \sum\nolimits_1 + O\,(x^2 \log^{2/3} x T_1), \text{ say.} \qquad \ldots(2.8)$$

Now, we have

$$\sum\nolimits_1 = \sum_{\delta \leqslant x} \frac{A_1(\delta t/(\delta, t))(\delta, t)}{\delta^2}$$

$$= \sum_{k \mid t} k \sum_{\substack{\delta \leqslant x \\ (\delta, t)=k}} \frac{A_1(\delta t/k)}{\delta^2}$$

$$= \sum_{k \mid t} k \sum_{\substack{m \leqslant x/k \\ (m, t/k)=1}} \frac{A_1(mt)}{m^2}. \qquad \ldots(2.9)$$

From (2.1) it is trivial to see that $A_1(m) \leqslant \tau(m)$, from which we infer that $\sum_{m=1}^{\infty} \frac{A_1(mt)}{m^2}$ converges for each fixed t. Also, we have

$$\sum_{m>x} \frac{A_1(mt)}{t^2} \leqslant \sum_{m>x} \frac{\tau(mt)}{m^2} \leqslant \tau(t) \sum_{m>x} \frac{\tau(m)}{m^2} = O(\tau(t)\, x^{-1} \log x). \qquad \ldots(2.10)$$

Now, from (2.10), (2.9), (2.5) and (2.4), we have

$$\sum\nolimits_1 = \sum_{k \mid t} \frac{1}{k} \left\{ \sum_{\substack{m=1 \\ (m, t/k)=1}}^{\infty} \frac{A_1(mt)}{m^2} + O\left(\sum_{m>x/k} \frac{A_1(mt)}{m^2} \right) \right\}$$

$$= A_2(t) + O\left(x^{-1} \log x \tau(t) \sum_{k \mid t} 1 \right)$$

$$= A_2(t) + O(x^{-1} \log x \tau^2(t)). \qquad \ldots(2.11)$$

Also, by (2.2) we have

$$\sum_{m \leqslant x} \frac{S(m)}{m} \leqslant \sum_{d\delta \leqslant x} \frac{2^{\omega(d)}}{\varphi(d)\, d\delta} = \sum_{d \leqslant x} \frac{2^{\omega(d)}}{\varphi(d)\, d} \sum_{\delta \leqslant x/d} \frac{1}{\delta}$$

$$= O\left(\log x \sum_{d \leqslant x} \frac{2^{\omega(d)}}{\varphi(d)\, d} \right) = O(\log x)$$

since

$$\sum_{d \leqslant x} \frac{2^{\omega(d)}}{\varphi(d)\, d} \leqslant \sum_{d \leqslant x} \frac{\tau(d)}{\varphi(d)\, d} \leqslant \sum_{d \leqslant x} \frac{\tau^2(d)}{d^2} = O(1).$$

Hence we have

$$\sum_{m \leqslant x} \frac{S(m)}{m} = O(\log x). \qquad \ldots(2.12)$$

From (2.2) it is easily seen that $S(mn) = S(m) \cdot S(n)/S((m, n))$. Since $S(m) \geqslant 1$ for all m, it follows that $S(mn) \leqslant S(m) \cdot S(n)$, for all m and n. Hence by (2.12) we have

$$T_1 = \sum_{\delta \leqslant x} \delta^{-1} S(\delta t/(\delta, t)) \leqslant S(t) \sum_{\delta \leqslant x} \delta^{-1} S(\delta) = O(S(t) \log x). \qquad \ldots(2.13)$$

Collecting now the results (2.8), (2.11) and (2.13), we obtain (2.3). Since $A_1(m) \leqslant \tau(m)$, we have by (2.5), $B_1(k; m) = O(\tau(m))$, so that by (2.4) we have $A_2(m) = O(\tau(m)) (\log m)$. Thus the last equality in (2.4) follows.

Hence Lemma 2.4 follows.

Remark 2.1 : From (2.4), (2.5) and (2.1), it is easily seen that

$$A_2(1) = B_1(1; 1) = \sum_{m=1}^{\infty} \frac{A_2(m)}{m^2} = \frac{\zeta(2)\,\zeta(3)}{\zeta(4)} = \frac{15\,\zeta(3)}{\pi^2}$$

since $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$. Hence by taking $t = 1$ in (2.3), we get

$$\sum_{m \leqslant x} \sigma^2(m) = \frac{5\zeta(3)\, x^3}{6} + O(x^2 \log^{5/3} x). \qquad \ldots(2.14)$$

Formula (2.14) has been established by Smith (1970, Theorem 2) in an entirely different way and was originally stated by Ramanujan [1916, eqn. (19)] with a weaker O-estimate, namely $O(x^2 \log^2 x)$.

3. Main Results

In this section we prove the following:

Theorem — Let r be a fixed integer $\geqslant 2$. Then for $x \geqslant 2$ we have

$$\sum_{\substack{m \leqslant x \\ t \mid m}} \sigma^r(m) = \frac{x^{r+1} \pi^2 A_r(t)}{6(r+1)\, t} + O(S(t)\, x^r \log^{(3r-1)/3} x), \qquad \ldots(3.1)$$

where the O-constant is independent of x and t,

$$A_r(m) = \sum_{k \mid m} \frac{B_{r-1}(k; m)}{k} = O(\tau(m) \log^{r-1} m) \qquad \ldots(3.2)$$

$$B_{r-1}(k; m) = \sum_{\substack{n=1 \\ (n, m/k)=1}}^{\infty} \frac{A_{r-1}(mn)}{n^2} \quad \text{for } k/m \qquad \ldots(3.3)$$

$A_1(m)$, $A_2(m)$ and $S(m)$ are given by (2.1), (2.4) and (2.2).

PROOF : We prove the theorem by induction on r. The theorem is true for $r = 2$ and all t, in virtue of Lemma 2.4. Assume the theorem for some $r \geqslant 2$ and all t. We have as in the proof of Lemma 2.4,

$$\sum_{\substack{m \leqslant x \\ t \mid m}} \sigma^{r+1}(m) = \sum_{\delta \leqslant x} \frac{1}{\delta} \sum_{\substack{m \leqslant x \\ \{\delta, t\} \mid m}} m\sigma^r(m). \qquad \ldots(3.4)$$

Now, by (3.1) and partial summation, it follows that

$$\sum_{\substack{m \leqslant x \\ t \mid m}} \sigma^r(m) = \frac{x^{r+2}\pi^2 A_r(t)}{6(r+2)\,t} + O\left(S(t)\, x^{r+1} \log^{(3r-1)/3} x\right). \qquad \ldots(3.5)$$

Making use of (3.5) in (3.4) we obtain

$$\begin{aligned}
\sum_{\substack{m \leqslant x \\ t \mid m}} \sigma^{r+1}(m) &= \sum_{\delta \leqslant x} \frac{1}{\delta} \left\{ \frac{x^{r+2}\pi^2 A_r(\{\delta, t\})}{6(r+2)\,\{\delta, t\}} \right. \\
&\qquad \left. + O\left(S(\{\delta, t\})\, x^{r+1} \log^{(3r-1)/3} x\right) \right\} \\
&= \frac{x^{r+2}\pi^2}{6(r+2)\,t} \sum_{\delta \leqslant x} \frac{A_r(\delta t/(\delta, t))\,(\delta, t)}{\delta^2} \\
&\qquad + O\left(x^{r+1} \log^{(3r-1)/3} x \sum_{\delta \leqslant x} \delta^{-1} S(\delta t/(\delta, t))\right) \\
&= \frac{x^{r+2}\pi^2}{6(r+2)\,t} \sum_{\delta \leqslant x} \frac{A_r(\delta t/(\delta, t))\,(\delta, t)}{\delta^2} \\
&\qquad + O\left(S(t)\, x^{r+1} \log^{(3r+2)/3} x\right),
\end{aligned} \qquad \ldots(3.6)$$

by (2.13). Now, as in the proof of (2.11) we can show that [by making use of the O-estimate for $A_r(m)$ given in (3.2)],

$$\begin{aligned}
&\sum_{\delta \leqslant x} \frac{A_r(\delta t/(\delta. t))\,(\delta, t)}{\delta^2} \\
&\qquad = A_{r+1}(t) + O\left(\tau^2(t) \log^{r-1} t \quad x^{-1} \log^r x\right)
\end{aligned} \qquad \ldots(3.7)$$

where $A_{r+1}(m)$ is the right side of (3·2) with r replaced by $r+1$. Substituting (3.7) in (3.6) we see that formula (3.1) is true for $r+1$. If $B_r(k; m)$ denotes the right side of (3.3) with r replaced by $r+1$, it follows from the O-estimate of $A_r(m)$ given in (3.2) that $B_r(k; m) = O(\tau(m) \log^{r-1} m)$. Hence we have

$$A_{r+1}(m) = \sum_{k \mid m} \frac{B_r(k; m)}{k} = O\left(\tau(m) \log^{r-1} m \sum_{k \mid m} \frac{1}{k}\right)$$

$$= O(\tau(m) \log^r m).$$

Thus (3.2) holds for $r+1$. Thus the induction is complete and hence the Theorem follows.

Taking $r = 3$ and $t = 1$ in the Theorem, after a tedious calculation, we get the following:

Corollary — For $x \geqslant 2$,

$$\sum_{m \leqslant x} \sigma^3(m) = \frac{5A\zeta(3)x^4}{8} + O(x^3 \log^{8/3} x) \quad \ldots(3.8)$$

where

$$A = \prod_p \left(1 + \frac{p^{12}+2p^{11}+p^{10}+p^9-3p^7-3p^6-3p^5-5p^4+2p^3+3p^2+2}{p(p^2+1)(p^3-1)(p^4-1)^2}\right).$$

Remark 3.1 : We note that the methods of this paper enable us to establish an asymptotic formula for the sum $\sum_{\substack{m \leqslant x \\ t \mid m}} \left(\frac{\sigma(m)}{m}\right)^r$ with error term $O(S(t) \log^{(3r-1)/3} x)$, where $S(t)$ is given by (2.2). An asymptotic formula for this sum in case $r = 2$ and $t = 1$ has been established by Smith (1970, Theorem 1) using different arguments.

Remark 3.2 : Mirsky (1949, Theorem 3) established an asymptotic formula for a more general sum, from which an asymptotic formula for the sum $\sum_{m \leqslant x} \left(\frac{\sigma(m)}{m}\right)^r$ can be deduced with an error term $O(\log^r x)$, which is weaker than the one mentioned above in Remark 3.1.

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