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FINSLER SPACES ADMITTING RECURRENT NEO-PSEUDO PROJECTIVE TENSOR FIELDS

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Riemannian spaces with recurrent curvature tensor field have been studied by Walker (1950), Ruse (1949) Finsler spaces with recurrent curvature tensor field have been studied by Mishra and Pande (1968), Sinha and Singh (1971) and many others. The object of the present paper is to define the recurrent neo-pseudo projective tensor field in Finsler space and discuss its properties.

1. INTRODUCTION

Let us consider an n-dimensional Finsler space F_n equipped with Berwald's connection coefficients $G^i_{jk}(x, \dot{x})$ (Rund 1959, pp. 76–79). This connection is used in defining the covariant derivative of a vector field $X^i(x, \dot{x})$, in the form

$$X^i_{(k)} = \partial_k X^i - \dot{\partial}_m X^i G^m_k + X^m G^i_{mk}{}^*. \qquad \text{...(1.1)}$$

It has been shown that the covariant derivative of the metric tensor $g_{ij}(x, \dot{x})$ is given by (Rund 1959, pp. 79–80)

$$g_{ij(k)} = -2A_{ijk\,|\,h}\, l^h, \qquad \text{...(1.2)}$$

where

$$A_{ikj} = FC_{ikj}, \quad C_{ikj} = \tfrac{1}{2}\frac{\partial g_{ij}}{\partial \dot{x}^k}.$$

The Berwald's deviation tensor field $H^j_k(x, \dot{x})$ (Rund 1959, pp. 124–26) and the projective deviation tensor field $W^j_k(x, \dot{x})$ (Rund 1959, pp. 137–41) are given by

$$H^j_k(x, \dot{x}) = 2\partial_k G^j - \dot{\partial}_k \partial_h G^j \dot{x}^h + 2G^j_{kl}\, G^l - \dot{\partial}_l G^j \dot{\partial}_k G^l \qquad \text{...(1.3)}$$

and

$$W^j_k(x, \dot{x}) = H^j_k - H\delta^j_k - (\dot{\partial}_i H^i_k - \dot{\partial}_k H)\frac{\dot{x}^j}{(n+1)}. \qquad \text{...(1.4)}$$

$^*\partial_m = \partial/\partial x^m$, $\dot{\partial}_m = \partial/\partial \dot{x}^m$.

These tensor fields $H^j_k(x, \dot{x})$ and $W^j_k(x, \dot{x})$ satisfy the following relations (Rund 1959, pp. 129, 140) :

$$\left.\begin{aligned} &\text{(a)} \quad H^j_k \dot{x}^k = 0 \\ &\text{(b)} \quad W^j_k \dot{x}^k = 0. \end{aligned}\right\} \qquad \ldots(1.5)$$

An H-recurrent Finsler space is characterized by (Mishra and Pande 1968)

$$H^i_{jkh(m)} = v_m H^i_{jkh}, \qquad \ldots(1.6)$$

where v_m is non-null recurrence vector field and $H^i_{jkh}(x, \dot{x})$ is Berwald's curvature tensor field.

The pseudo-deviation tensor field $T^j_k(x, \dot{x})$ defined by Sinha (1971) is given by

$$T^j_k(x, \dot{x}) = -\left[H\delta^j_k + \frac{\dot{x}^j}{(n+1)}(\dot{\partial}_i H^i_k - \dot{\partial}_k H)\right]. \qquad \ldots(1.7)$$

We have also (Sinha 1971)

$$\text{(a)} \quad T^j_k \dot{x}^k = 0, \quad \text{(b)} \; T_h \dot{x}^h = (n-1)\, T \qquad \ldots(1.8)$$

where

$$T^i_i = (n-1)\, T.$$

The following commutation formulae will be used in the sequel (Rund 1959, pp. 126–27)

$$\left.\begin{aligned} &\text{(a)} \quad T^i_{j(h)(k)} - T^i_{j(k)(h)} = -(\dot{\partial}_r T^i_j) H^r_{hk} - T^i_r H^r_{jhk} + T^r_j H^i_{rhk} \\ &\text{(b)} \quad (\dot{\partial}_k T^i_j)_{(h)} - \dot{\partial}_k T^i_{j(h)} = T^i_r G^r_{jkh} - T^r_j G^i_{rkh} \end{aligned}\right\} \qquad \ldots(1.9)$$

where $T^i_j(x, \dot{x})$ used here is any tensor field.

The neo-pseudo projective tensor field $Q^i_j(x, \dot{x})$ (Singh and Singh 1979) is defined by

$$Q^i_j \overset{def}{=} pW^i_j + qT^i_j \qquad \ldots(1.10)$$

where p and q are scalar functions of $(x, \dot{x})$ and homogeneous of degree zero in its directional arguments.

The neo-pseudo projective curvature tensor fields $Q^i_{hj}(x, \dot{x})$ and $Q^i_{lhj}(x, \dot{x})$ are defined by

$$\text{(a)} \quad Q^i_{hj} = \tfrac{2}{3}\dot{\partial}_{[h}Q^i_{j]}, \quad \text{(b)} \quad Q^i_{lhj} = \dot{\partial}_l Q^i_{hj} = \tfrac{2}{3}\dot{\partial}^2_{l\,[h}Q^i_{j]}, \qquad \ldots(1.11)$$

where $[h, j]$ stands for skew-symmetric part in h, j.

The tensor fields $Q^i_{hj}(x, \dot{x})$ and $Q^i_{lhj}(x, \dot{x})$ may be expressed as (Singh and Singh 1979)

$$\text{(a)} \quad Q^i_{hj} = pW^i_{hj} + qT^i_{hj} + \tfrac{2}{3}[\dot{\partial}_{[h}pW^i_{j]} + \dot{\partial}_{[h}qT^i_{j]}]$$

$$\text{(b)} \quad Q^i_{lhj} = pW^i_{lhj} + qT^i_{lhj} + (\dot{\partial}_l p)\, W^i_{hj} + (\dot{\partial}_l q)\, T^i_{hj}$$

$$+ \tfrac{2}{3}[\dot{\partial}^2_{l\,[h}PW^i_{j]} + \dot{\partial}_{[h}P\dot{\partial}_{|l|}W^i_{j]} + \dot{\partial}^2_{l\,[h}qT^i_{j]} + \dot{\partial}_{[h}q\dot{\partial}_{|l|}T^i_{j]}]^*. \qquad \ldots(1.12)$$

We have the following identities (Singh and Singh 1979):

$$\text{(a)} \quad Q^i_{hj}\,\dot{x}^h = Q^i_j \quad \text{(b)} \quad Q^i_{lhj}\,\dot{x}^l = Q^i_{hj} \qquad \ldots(1.13)$$

and

$$\left.\begin{aligned} &\text{(a)} \quad Q^i_i = q(n-1)\,T \\ &\text{(b)} \quad Q^i_{hi} = qT_h + \tfrac{1}{3}[(\dot{\partial}_h q)(n-1)\,T - (\dot{\partial}_i p)\,W^i_h - (\dot{\partial}_i q)\,T^i_h\,]. \end{aligned}\right\} \qquad \ldots(1.14)$$

2. Q-Recurrent Finsler Spaces in the Sense of Berwald's Covariant Derivative

Definition 2.1 — An n-dimensional Finsler space F_n is said to be a Q-recurrent Finsler space, if the Berwald's covariant derivative of $Q^i_{lhj}(x, \dot{x})$ satisfies the relation

$$Q^i_{lhj(k)} = \lambda_k Q^i_{lhj} \qquad \ldots(2.1)$$

where $\lambda_k(x, x)$ is non-null recurrence vector field.

Transvecting (2.1) by $\dot{x}^l$ and using (1.13b), we have

$$Q^i_{hj(k)} = \lambda_k Q^i_{hj}. \qquad \ldots(2.2)$$

Again multiplying (2.2) by $\dot{x}^h$ and noting (1.13a), we have

$$Q^i_{j(k)} = \lambda_k Q^i_j\,. \qquad \ldots(2.3)$$

*The index in " | | " is free from skew-symmetric part.

Hence, from (2.2) and (2.3) we conclude that the tensor fields $Q^i_{hj}(x, \dot{x})$ and $Q^i_j(x, \dot{x})$ are also recurrent in a Q-recurrent Finsler space.

Theorem 2.1 — The recurrence vector of Q^i_j-recurrent Finsler space satisfies the relation

$$2\lambda_{[k(m)]} = H^r_{mk}\, \dot{\partial}_r(\log qT). \qquad \ldots(2.4)$$

PROOF : Differentiating (2.3) covariantly with respect to x^m, we have

$$Q^i_{j(k)(m)} = (\lambda_{k(m)} + \lambda_k\lambda_m)\, Q^i_j\,. \qquad \ldots(2.5)$$

Commutating the indices k and m in (2.5) and applying the commutation formula (1.9a), we have

$$-(\dot{\partial}_r Q^i_j)\, H^r_{km} - Q^i_r\, H^r_{jkm} + Q^r_j\, H^i_{rkm} = 2\lambda_{[k(m)]}Q^i_j\,.$$

Contracting the above equation with respect to the indices i and j, and using (1.14a), we have

$$2\lambda_{[k(m)]}qT = -\,\dot{\partial}_r(qT)\, H^r_{km}\,. \qquad \ldots(2.6)$$

Since $H^r_{km} = -H^r_{mk}$, hence eqn. (2.6) reduces to (2.4).

Equation (2.2) shows that in a Q-recurrent Finsler space the tensor field $Q^i_{hj}(x, \dot{x})$ is recurrent. But the converse of this is not, in general, true. In the following theorem we shall find a condition under which the converse will hold.

Theorem 2.2 — If the tensor field $Q^i_{hj}(x, \dot{x})$ is recurrent in F_n then it is a Q-recurrent in F_n provided that the recurrence vector λ_k satisfies the relation

$$(\dot{\partial}_l\lambda_m)\, q(n-1)\, T = 0. \qquad \ldots(2.7)$$

PROOF : In view of commutation formula (1.9b) for the tensor field $Q^i_{hj}(x, \dot{x})$, we have

$$(\dot{\partial}_l Q^i_{hj})_{(m)} - \dot{\partial}_l Q^i_{hj(m)} = Q^i_{rj}\, G^r_{hlm} + Q^i_{hr}\, G^r_{jlm} - Q^r_{hj}\, G^i_{rlm}. \qquad \ldots(2.8)$$

Differentiating (2.2) partially with respect to $\dot{x}^l$ and applying the equation (2.8) we have

$$(\dot{\partial}_l Q^i_{hj})_{(m)} - (\dot{\partial}_l\lambda_m)\, Q^i_{hj} - \lambda_m\dot{\partial}_l\, Q^i_{hj} = Q^i_{rj}\, G^r_{hlm} + Q^i_{hr}\, G^r_{jlm} - Q^r_{hj}\, G^i_{rlm}.$$

Contracting the above equation with respect to the indices i and j, and applying (2.1), we have

$$(\dot{\partial}_l \lambda_m)\, Q^i_{hi} = -Q^i_{ri}\, G^r_{hlm}.$$

Multiplying the last equation by $\dot{x}^h$, using the relation (1.14b) and $G^r_{hlm}\, \dot{x}^h = 0$, we have

$$(\dot{\partial}_l \lambda_m)\,[qT_h + \tfrac{1}{3}\{(\dot{\partial}_h q)\,(n-1)\,T - (\dot{\partial}_i p)\, W^i_h - (\dot{\partial}_i q)\; T^i_h\}]\, \dot{x}^h = 0.$$

In view of eqns. (1.5b), (1.8) and the relation $(\dot{\partial}_h q)\dot{x}^h = 0$ the above equation yields the required result.

Theorem 2.3 — If Q^i_{hj}-recurrent Finsler space is projectively flat then the following relation holds:

$$\dot{x}^k\{pW^i_{lhj(k)} + q_{(k)}\, T^i_{lhj} + q(T^i_{lhj(k)} - \lambda_k T^i_{lhj}) + N^i_{lhj(k)} - \lambda_k N^i_{lhj}\}$$

$$= \dot{x}^k(\dot{\partial}_l \lambda_k)\, Q^i_{hj} \qquad \ldots(2.9)$$

where

$$N^i_{lhj}\,(x, \dot{x}) \overset{def}{=} (\dot{\partial}_l p)\; W^i_{hj} + (\dot{\partial}_l q)\; T^i_{hj} + \tfrac{2}{3}\{\dot{\partial}^2_l\,{}_{[h} p W^i_{j]}$$

$$+ \dot{\partial}_{[h} p \dot{\partial}_{|l|} W^i_{j]} + \dot{\partial}^2_l\,{}_{[h} q T^i_{j]} + \dot{\partial}_{[h} q \dot{\partial}_{|l|} T^i_{j]}\}. \qquad \ldots(2.10)$$

PROOF : By Theorem 2.2, we have

$$(\dot{\partial}_l Q^i_{hj})_{(k)} - (\dot{\partial}_l \lambda_k)\, Q^i_{hj} - \lambda_k \dot{\partial}_l Q^i_{jh} = Q^i_{rj}\; G^r_{hlk} + Q^i_{hr} G^r_{jlk} - Q^r_{hj}\; G^i_{rlk}.$$

Multiplying this equation by $\dot{x}^k$ and using $G^i_{rlk}\; \dot{x}^k = 0$, the relation (1.11b), we have

$$\dot{x}^k(\dot{\partial}_l \lambda_k)\, Q^i_{hj} = (Q^i_{lhj(k)} - \lambda_k Q^i_{lhj})\; \dot{x}^k.$$

Substituting the value of $Q^i_{lhj}\,(x, \dot{x})$ from (1.12b) in above equation and making use of (2.10), we have

$$\dot{x}^k(\dot{\partial}_l \lambda_k)\, Q^i_{hj} = [\{pW^i_{lhj} + qT^i_{lhj} + N^i_{lhj}\}_{(k)} - \lambda_k\{pW^i_{lhj}$$

$$+ qT^i_{lhj} + N^i_{lhj}\}]\; \dot{x}^k. \qquad \ldots(2.11)$$

Since the space is projectively flat, that is $W^i_{lhj} = 0$, then (2.11) yields (2.9).

Theorem 2.4 — If Q^i_{hj}-recurrent Finsler space is projectively flat and $G^i_{lhj} = 0$, then the following relation is true:

$$\dot{x}^k\{q_{(k)}T^i_{lhj} + q(v_k - \lambda_k)\,T^i_{lhj} + N^i_{lhj(k)} - \lambda_k N^i_{lhj}\} = \dot{x}^k(\dot{\partial}_l\lambda_k)\,Q^i_{hj}. \qquad ...(2.12)$$

PROOF : When $G^i_{lhj} = 0$, then it has been shown by Sinha and Singh (1971) that projective curvature tensor field $W^i_{lhj}(x, \dot{x})$ is recurrent with the same non-null recurrence vector field v_k as the curvature tensor field $H^i_{jkh}(x, \dot{x})$. Hence we have the relation

$$W^i_{lhj(k)} = v_k W^i_{lhj}. \qquad ...(2.13)$$

Equations (1.6), (2.13) and the relation $T^i_{lhj} = W^i_{lhj} - H^i_{lhj}$ (defined by Sinha 1971) show that

$$T^i_{lhj(k)} = v_k T^i_{lhj}. \qquad ...(2.14)$$

In view of eqns. (2.13), (2.14) and $W^i_{lhj} = 0$, eqn. (2.9) reduces to (2.12).

Theorem 2.5 — In an n-dimensional Q-recurrent Finsler space the following relation holds:

$$\dot{x}^k\{Q^i_{hj}\,\dot{\partial}_l\lambda_{k(s)} + Q^i_{jl}\,\dot{\partial}_h\lambda_{k(s)} + Q^i_{lh}\,\dot{\partial}_j\lambda_{k(s)}\} = 0. \qquad ...(2.15)$$

PROOF : By Theorem 2.2, we have

$$(\dot{\partial}_l Q^i_{hj})_{(k)} - (\dot{\partial}_l\lambda_k)\,Q^i_{hj} - \lambda_k\dot{\partial}_l Q^i_{hj} = Q^i_{rj}\,G^r_{hlk} + Q^i_{hr}\,G^r_{jlk} - Q^r_{hj}\,G^i_{rlk}$$

with the help of (2.1), the above equation reduces to

$$(\dot{\partial}_l\lambda_k)\,Q^i_{hj} = Q^r_{hj}\,G^i_{rlk} - Q^i_{rj}\,G^r_{hlk} - Q^i_{hr}\,G^r_{jlk}. \qquad ...(2.16)$$

Cyclic permutation of the indices l, h and j in (2.16) yields two more results. On adding these two relations with (2.16), we have

$$(\dot{\partial}_l\lambda_k)\,Q^i_{hj} + (\dot{\partial}_h\lambda_k)\,Q^i_{jl} + (\dot{\partial}_j\lambda_k)\,Q^i_{lh} = Q^r_{hj}\,G^i_{rlk} + Q^r_{jl}\,G^i_{rhk} + Q^r_{lh}\,G^i_{rjk}. \qquad ...(2.17)$$

Differentiating (2.17) covariantly with respect to x^s and applying the commutation formula $(\dot{\partial}_k X^i)_{(h)} = \dot{\partial}_k X^i_{(h)} - X^r G^i_{rkh}$, the relation (2.16), we have

$$(\dot{\partial}_l \lambda_{k(s)} + \lambda_r G^r_{kls}) Q^i_{hj} + (\dot{\partial}_h \lambda_{k(s)} + \lambda_r G^r_{khs}) Q^i_{jl} + (\dot{\partial}_j \lambda_{k(s)} + \lambda_r G^r_{kjs}) Q^i_{lh}$$

$$= Q^r_{hj} G^i_{rlk(s)} + Q^r_{jl} G^i_{rhk(s)} + Q^r_{lh} G^i_{rjk(s)}. \qquad \ldots(2.18)$$

Multiplying (2.18) by $\dot{x}^k$ and using $G^i_{rlk} \dot{x}^k = 0$, $\dot{x}^k_{(s)} = 0$, we get the required result.

References

Mishra, R. S., and Pande, H. D. (1968). Recurrent Finsler spaces. *J. Indian Math. Soc.*, **32**, 17–22.

Rund, H. (1959). The Differential Geometry of Finsler Spaces. Springer-Verlag, Berlin.

Ruse, H. S. (1949). Three dimensional spaces of recurrent curvature. *Proc. Lond. math. Soc.*, **50**(2), 438–41.

Singh, U. P., and Singh, A. K. (1979). On neo-pseudo-projective tensor field. *Indian J. pure appl. Math.*, **10**, 1196–1201.

Sinha, B. B. (1971). On projectively flat Finsler space and pseudo-deviation tensor field. *Prog. Math.*, **5**, 87–92.

Sinha, B. B., and Singh, S. P. (1971). On recurrent Finsler spaces. *Rev. Roum. Math. Pures Appl.*, **16**, 977–86.

Walker, A. G. (1950). On Ruse's spaces of recurrent curvature. *Proc. Lond. math. Soc.*, **52**(2), 36–64.