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ON THE N-CURVATURE COLLINEATION IN FINSLER SPACE II

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In our earlier paper (Singh and Singh 1981) we have defined N-curvature collineation and discussed the existence of N-curvature collineation of several types. The purpose of the present paper is to discuss the relation between N-curvature collineation and other symmetries admitted by an affinely connected space.

1. INTRODUCTION

Let F_n be an n-dimensional Finsler space equipped with $2n$ line elements $(x^i, \dot{x}^i)$ and positively homogeneous metric function $F(x, \dot{x})$ of degree one in directional arguments $\dot{x}^i$.

The covariant derivative of any vector field $X^i(x, \dot{x})$ with respect to x^j is given by (Yano 1957, p. 196)

$$\nabla_j X^i = \partial_j X^i - \dot{\partial}_m X^i \, \pi^{\;m}_{hj} \, \dot{x}^h + X^m \pi^{\;i}_{mj} \qquad \ldots(1.1)$$

where

$$\pi^{\;i}_{jk} = G^{\;i}_{jk} - \frac{1}{n+1} \dot{\partial}_j G^{\;r}_{kr} \, \dot{x}^i \qquad \ldots(1.2)$$

is projective normal connection coefficients and satisfy the following relation (Yano 1957, p. 196)

$$\text{(a)} \;\; \dot{\partial}_l \;\, \pi^{\;i}_{jk} \, \dot{x}^l = 0, \qquad \text{(b)} \;\; \pi^{\;i}_{jk} = \pi^{\;i}_{kj}. \qquad \ldots(1.3)$$

The symbol $G^{\;i}_{jk}(x, \dot{x}) = \dot{\partial}_j \dot{\partial}_k G^i(x, \dot{x})$, given in (1.2), are Berwald's connection parameters (Rund 1959, p. 71).

We have the following commutation formulae involving the Lie-derivatives of any tensor field $T^{\;i}_{jk}(x, \dot{x})$ and the connection coefficients $\pi^{\;i}_{jk}(x, \dot{x})$ (Yano 1957, p. 201):

$$\left.\begin{aligned}
&\text{(a)} \quad \dot{\partial}_l(\underset{v}{\mathcal{L}} T^{\;i}_{jk}) - \underset{v}{\mathcal{L}}(\dot{\partial}_l T^{\;i}_{jk}) = 0, \\
&\text{(b)} \quad \underset{v}{\mathcal{L}}(\nabla_l T^{\;i}_{jk}) - \nabla_l(\underset{v}{\mathcal{L}} T^{\;i}_{jk}) = T^{\;a}_{jk} \underset{v}{\mathcal{L}} \pi^{\;i}_{al} - T^{\;i}_{ak} \underset{v}{\mathcal{L}} \pi^{\;a}_{jl} \\
&\qquad\qquad - T^{\;i}_{ja} \underset{v}{\mathcal{L}} \pi^{\;a}_{kl} - (\dot{\partial}_a T^{\;i}_{jk}) (\underset{v}{\mathcal{L}} \pi^{\;a}_{bl}) \, \dot{x}^b,
\end{aligned}\right\} \qquad \ldots(1.4)$$

$$\nabla_j(\underset{v}{\mathcal{L}}\pi^i_{kh}) - \nabla_k(\underset{v}{\mathcal{L}}\pi^i_{jh}) = \underset{v}{\mathcal{L}}N^i_{jkh} + (\dot{\partial}_l\pi^i_{kh})(\underset{v}{\mathcal{L}}\pi^l_{bj})\dot{x}^b - (\dot{\partial}_l\pi^i_{jh})(\underset{v}{\mathcal{L}}\pi^l_{bk})\dot{x}^b \qquad \ldots(1.5)$$

where $N^i_{jkh}(x, \dot{x})$ is the normal projective curvature tensor field and satisfies the following (Yano 1957, p. 196)

$$\left.\begin{array}{lll} \text{(a)} & N_{kh} \overset{def}{=} N^i_{ikh}, & \text{(b)} \quad \dot{\partial}_r N^i_{jkh}\,\dot{x}^r = 0 \\ \text{and (c)} & N^i_{jkh} = -N^i_{kjh}. & \end{array}\right\} \qquad \ldots(1.6)$$

We shall quote the following definitions for reference in the latter articles of this paper:

N-Curvature Collineation (Singh and Singh 1981) — A Finsler space F_n is said to admit N-curvature collineation if there exists a vector field v^i such that

$$\underset{v}{\mathcal{L}}N^i_{jkh} = 0. \qquad \ldots(1.7)$$

N-Ricci Collineation (Singh and Singh 1981) — A Finsler space F_n is said to admit N-Ricci collineation if there exist a vector field v^i such that

$$\underset{v}{\mathcal{L}}N_{kh} = 0. \qquad \ldots(1.8)$$

Affine Motion (Singh and Singh 1981) — A Finsler space F_n is said to admit an affine motion provided there exists a vector field v^i such that

$$\underset{v}{\mathcal{L}}\pi^i_{jk} = 0. \qquad \ldots(1.9)$$

2. Relation Between Projective Motion and N-Curvature Collineation

Let us consider an infinitesimal point transformation

$$\bar{x}^i = x^i + v^i(x)\,d\tau \qquad \ldots(2.1)$$

where $v^i(x)$ is a vector field independent of the directional arguments $\dot{x}^i$ and $d\tau$ is an infinitesimal point constant.

If the infinitesimal point transformation (2.1) transforms the system of geodesics into the same system, (2.1) is called an infinitesimal projective motion. The necessary and sufficient condition that (2.1) be a projective motion in F_n is that the Lie-derivative of $\pi^i_{jk}(x, \dot{x})$ with respect to (2.1) has the form (Yano 1957, p. 200)

$$\underset{v}{\mathcal{L}}\pi^i_{jk} = \delta^i_j\, p_k + \delta^i_k\, p_j \qquad \ldots(2.2)$$

where $p(x, \dot{x})$ is an arbitrary scalar function positively homogeneous of degree one in $\dot{x}^i$ and satisfies the following relations

$$p_k \dot{x}^k = p, \text{ as } p_k = \dot{\partial}_k p. \quad \ldots(2.3)$$

An affinely connected Finsler space is characterised by relation (Rund 1959, p. 81)

$$\dot{\partial}_j G^i_{kh} = 0. \quad \ldots(2.4)$$

The relations (2.4) and (1.2), for affinely connected Finsler space F_n, give

$$\dot{\partial}_j \pi^i_{kh} = 0. \quad \ldots(2.5)$$

Making use of the relations (2.2) and (2.5) in formula (1.5), we have

$$\underset{v}{\mathcal{L}} N^i_{jkh} = \delta^i_h (\nabla_j p_k - \nabla_k p_j) + \delta^i_k \nabla_j p_h - \delta^i_j \nabla_k p_h \quad \ldots(2.6)$$

where we have used the fact that covariant derivative of δ^i_k vanishes.

If the projective motion, for an affinely connected Finsler space F_n, is N-curvature collineation, we have, from (1.7) and (2.6)

$$\delta^i_h (\nabla_j p_k - \nabla_k p_j) + \delta^i_k \nabla_j p_h - \delta^i_j \nabla_k p_h = 0. \quad \ldots(2.7)$$

Contracting the above relation with respect to the indices i and j, we have

$$\nabla_h p_k - n \nabla_k p_h = 0. \quad \ldots(2.8)$$

Interchanging the indices h and k in eqn. (2.8), we get

$$\nabla_k p_h - n \nabla_h p_k = 0. \quad \ldots(2.9)$$

Substituting the value of $\nabla_k p_h$ from (2.9) in relation (2.8) we get $(n^2 - 1) \nabla_h p_k = 0$ which gives

$$\nabla_h p_k = 0 \text{ for } n > 1. \quad \ldots(2.10)$$

Conversely, if $\nabla_h p_k = 0$, then the relation (2.6) gives $\underset{v}{\mathcal{L}} N^i_{jkh} = 0$. Hence we have the following

Theorem 2.1 — In an affinely connected Finsler space the necessary and sufficient condition that a projective motion be N-curvature collineation is that p_i is covariant constant.

Now, contracting (2.6) with respect to the indices i, j and using the relation (1.6*a*), we get

$$\underset{v}{\mathcal{L}} N_{kh} = \nabla_h p_k - n \nabla_k p_h. \quad \ldots(2.11)$$

If the projective motion in an affinely connected Finsler space is N-Ricci collineation then $\underset{v}{\mathcal{L}} N_{kh} = 0$ which in view of (2.11) after simplification leads to

$$\nabla_h p_k = 0 \text{ for } n > 1. \quad \ldots(2.12)$$

Conversely, if $\nabla_h p_k = 0$ then the relation (2.11) gives $\underset{v}{\mathcal{L}} N_{kh} = 0$. Hence we have the following theorem:

Theorem 2.2 — In an affinely connected Finsler space the necessary and sufficient condition that a projective motion will be N-Ricci collineation is that p_i is covariant constant.

It may be noted that in Finsler space a N-Ricci collineation is not, in general, a N-curvature collineation even if the space is affinely connected. However, Theorems 2.1 and 2.2 prove the following:

Theorem 2.3 — In an affinely connected Finsler space a N-Ricci collineation is a N-curvature collineation provided that it is a projective motion.

3. NP-Symmetric Finsler Space (NP-SF_n)

A Finsler space F_n is said to be normal projective symmetric Finsler space if the normal projective curvature tensor field $N^i_{jkh}(x, \dot{x})$ satisfies the relation.

$$\nabla_l N^i_{jkh} = 0. \quad \ldots(3.1)$$

The normal projective symmetric Finsler space is denoted by NP-SF_n.

Contraction of relation (3.1) with respect to the indices i and j yields

$$\nabla_l N_{kh} = 0, \quad \ldots(3.2)$$

where we have used the relation (1.6a). In this case the Finsler space is called normal projective Ricci-symmetric Finsler space.

Applying the commutation formula (1.4b) for the normal projective curvature tensor field $N^i_{jkh}(x, \dot{x})$ and using the relations (1.6b), (2.2), (2.3) and (3.1), we have

$$\nabla_l(\underset{v}{\mathcal{L}} N^i_{jkh}) = \dot{\partial}_l N^i_{jkh}\, p + 2N^i_{jkh}\, p_l - p_a N^a_{jkh}\, \delta^i_l + N^i_{lkh}\, p_j$$
$$+ N^i_{jlh}\, p_k + N^i_{jkl}\, p_h. \quad \ldots(3.3)$$

Transvecting eqn. (3.3) by $\dot{x}^k$, $\dot{x}^h$, $\dot{x}^l$ and using the relations (1.6b), (2.3) we have

$$\nabla_l(\underset{v}{\mathcal{L}} N^i_{jkh})\, \dot{x}^k \dot{x}^h \dot{x}^l = 4N^i_{jkh}\, \dot{x}^k \dot{x}^h p - p_a N^a_{jkh}\, \dot{x}^k \dot{x}^h \dot{x}^i$$
$$+ N^i_{lkh}\, \dot{x}^k \dot{x}^h \dot{x}^l p_j. \quad \ldots(3.4)$$

In view of (1.7) the above relation reduces to

$$4N^{i}_{jkh}\,\dot{x}^k\dot{x}^h p - p_a N^{a}_{jkh}\,\dot{x}^k\dot{x}^h\dot{x}^i + N^{i}_{lkh}\,\dot{x}^k\dot{x}^h\dot{x}^l\, p_j = 0. \qquad \ldots(3.5)$$

Contracting (3.5) with respect to the indices i and j, we get

$$N_{kh}\,\dot{x}^k\dot{x}^h\, p = 0 \qquad \ldots(3.6)$$

from which it follows that either $N_{kh}\dot{x}^k\dot{x}^h = 0$ or $p = 0$. If $p = 0$ then $p_i = 0$ which in view of relation (2.2) gives $\underset{v}{\mathcal{L}}\pi^{i}_{jk} = 0$. Hence, we have the following theorem:

Theorem 3.1 — If, in a normal projective symmetric Finsler space, the projective motion is N-curvature collineation then either it is affine motion or $N_{kh}\dot{x}^k\dot{x}^h = 0$.

Contracting (3.3) with respect to the indices i, j and using the relation (1.6), we have

$$\nabla_l(\underset{v}{\mathcal{L}}N_{kh}) = \dot{\partial}_l N_{kh} p + 2N_{kh}p_l + N_{lh}p_k + N_{kl}p_h. \qquad \ldots(3.7)$$

In view of relation (1.8), the above relation reduces to

$$\dot{\partial}_l N_{kh}\, p + 2N_{kh}p_l + N_{lh}p_k + N_{kl}p_h = 0. \qquad \ldots(3.8)$$

Transvecting eqn. (3.8) by $\dot{x}^k\dot{x}^h\dot{x}^l$ and using the relation (2.3), we get

$$N_{kh}\dot{x}^k\dot{x}^h p = 0, \qquad \ldots(3.9)$$

where we have also used the relation $\dot{\partial}_l N_{kh}\dot{x}^l = 0$.

Equation (3.9) gives either $N_{kh}\dot{x}^k\dot{x}^h = 0$ or $p = 0$.

If $p = 0$ then $p_i = 0$ which in view of the relation (2.2) gives to $\underset{v}{\mathcal{L}}\pi^{i}_{jk} = 0$. Hence, we have the following theorem:

Theorem 3.2 — If, in a normal projective symmetric Finsler space, the projective motion is N-Ricci collineation then either it is affine motion or $N_{kh}\dot{x}^k\dot{x}^h = 0$.

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