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ON SUM AND PRODUCT OF NORMAL OPERATORS

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It is known that if each of two normal operators commutes with the adjoint of the other, then their sum and product are normal In this note weaker conditions are discussed, which ensure normality (hyponormality) of the sum and product of normal (hyponormal) operators. Also an observation is made about operators T for which T^*T commutes with $T + T^*$.

All operators in this note are bounded linear transformations on a Hilbert space H. An operator A is normal (hyponormal) if $AA^* = A^*A(AA^* \leqslant A^*A)$.

Yadav and Ramanujan (1967, 1969) have proved that if the real part of each of two normal (hyponormal) operators commutes with the imaginary part of the other, then their sum is normal (hyponormal). The purpose of this note is to discuss the normality (hyponormality) of the product of two normal (hyponormal) operators.

First we state without proofs the following results which are easy applications of Putnam-Fuglede theorem [Problem 152 of Halmos (1967)].

Proposition 1 — Let $T_j = A_j + iB_j (j = 1, 2)$ be operators in their cartesian form and suppose that T_1 is normal. Then the following are equivalent:

(i) $T_1 T_2^* = T_2^* T_1$

(ii) each of A_1, B_1 commutes with A_2, B_2.

Proposition 2 — Let N_1, N_2 be normal operators such that $N_1 N_2^*$ and $N_2^* N_1$ are self-adjoint. Then $N_1 + N_2$ is normal if and only if N_1 commutes with N_2.

Also we note that the result in Yadav and Ramanujan (1969) is immediate in view of the observation that given normal operators $N_j = A_j + iB_j (j = 1, 2)$ in their cartesian form, $N_1 + N_2$ is normal if and only if

$$(A_1 B_2 - B_2 A_1) + (A_2 B_1 - B_1 A_2) = 0.$$

Remark : In Proposition 2, the self-adjointness of $N_1N_2^*$ and $N_2^* N_1$ cannot be omitted. To see this, let $N_1 = UU^*$ and $N_2 = U + U^*$, where U is the unilateral shift.

It can be easily seen that if T_1 and T_2 are hyponormal operators such that T_1 commutes with T_2^*, then $T_1 + T_2$ is hyponormal. However it is shown in Abrahamse (1978) that if T_1 and T_2 are commuting hyponormal operators, then $T_1 + T_2$ need not be hyponormal.

It is well-known that the product of two hyponormal operators need not be hyponormal even if they commute [Problem 164 of Halmos (1967)]. However if we consider some other commutativity conditions, then the product turns out to be hyponormal as shown in the following theorem.

Recall that the positive part of an operator A is $(A^*A)^{1/2}$.

Theorem 3 — Let T_1 and T_2 be hyponormal operators, suppose that T_1 commutes with the positive part of T_2 and T_2 commutes with the positive part of T_1^*. Then T_1T_2 and T_2T_1 are hyponormal.

PROOF : First we prove hyponormality of T_1T_2. By hypothesis

$$T_1^* (T_2^* T_2) = (T_2^* T_2) T_1^* \text{ and } T_2^* (T_1 T_1^*) = (T_1T_1^*) T_2^*.$$

Since for a positive operator P, R^*PR is a positive operator for every operator R, we have

$$\begin{aligned}(T_1T_2)^* (T_1T_2) - (T_1T_2) (T_1T_2)^* &= T_2^* T_1^* T_1T_2 - T_1T_2T_2^* T_1^* \\ &\geqslant T_2^* T_1T_1^* T_2 - T_1T_2T_2^* T_1^* \\ &\geqslant T_1T_1^* T_2^* T_2 - T_1T_2^* T_2T_1^* \\ &= T_1T_1^* T_2^*T_2 - T_1T_1^* T_2^* T_2 \\ &= 0.\end{aligned}$$

Next we establish the hyponormality of T_2T_1. We note that for any operator A and $x \in H$, $\|Ax\|^2 = (Ax, Ax) = (A^*Ax, x) = ((A^*A)^{1/2} x, (A^*A)^{1/2} x) = \|(A^*A)^{1/2} x\|^2$. Therefore,

$$\begin{aligned}\|(T_2T_1)^*x\| &= \| T_1^* T_2^* x \| \\ &= \| (T_1T_1^*)^{1/2} T_2^* x \|\end{aligned}$$

(*equation continued on p.* 1215)

$$= \| T_2^* (T_1 T_1^*)^{1/2} x \|$$

$$\leqslant \| T_2 (T_1 T_1^*)^{1/2} x \|$$

$$= \| (T_2^* T_2)^{1/2} (T_1 T_1^*)^{1/2} x \|$$

$$= \| T_1^* (T_2^* T_2)^{1/2} x \|$$

$$\leqslant \| T_1 (T_2^* T_2)^{1/2} x \|$$

$$= \| T_2 T_1 x \|.$$

Remark : The conclusion of the theorem does not hold in the absence of the condition of T_2 commuting with positive part of T_1^*, as shown in the following example.

Let H be a separable Hilbert space with orthonormal basis $\{e_n\}_{n=0}^{\infty}$. Let T_1 be the orthogonal projection of H onto $[e_0, e_1]$, the linear span of $\{e_0, e_1\}$ and T_2 be defined as $T_2 e_n = e_{n+1}$ $(n \geqslant 0)$. Then T_1, T_2 are hyponormal operators and T_1 commutes with $T_2^* T_2$. But

$$(T_1 T_2)^* e_1 = T_2^* T_1^* e_1 = T_2^* e_1 = e_0 \text{ and } T_1 T_2 e_1 = T_1 e_2 = 0.$$

Therefore $\| (T_1 T_2)^* e_1 \| \nleqslant \| (T_1 T_2) e_1 \|$. Hence $T_1 T_2$ is not hyponormal.

Corollary 4 — Let N_1 and N_2 be normal operators. Then each of N_1 and N_2 commutes with the positive part of the other if and only if $N_1 N_2$ and $N_2 N_1$ are normal.

PROOF : Since the positive part of N_1 and N_2 are same as N_1^* and N_2^* respectively, it follows from Theorem 3 that $N_1 N_2$ and $(N_1 N_2)^* = N_2^* N_1^*$ are both hyponormal operators. This gives the normality of $N_1 N_2$. The normality of $N_2 N_1$ follows similarly.

The converse part is a consequence of the following result of Kaplansky (1953) : Let A and B be operators on a Hilbert space such that A and AB are normal. Then B commutes with $A^* A$ if and only if BA is normal.

Corollary 5 — Let $N_j = U_j P_j (j = 1, 2)$ be normal operators in their polar decomposition, suppose that $U_1 P_2 = P_2 U_1$, $U_2 P_1 = P_1 U_2$ and $P_1 P_2 = P_2 P_1$. Then $N_1 N_2$ and $N_2 N_1$ are normal.

PROOF : $N_1P_2 = U_1P_1P_2 = U_1P_2P_1 = P_2U_1P_1 = P_2N_1$. Hence N_1 commutes with the positive part of N_2. Similarly N_2 commutes with the positive part of N_1. Thus N_1N_2 and N_2N_1 are normal.

Remark : The condition that P_1 commutes with P_2 cannot be despensed with in Corollary 5. In fact for self-adjoint operators P and Q with P positive, the following are equivalent:

(i) $PQ = QP$

(ii) PQ is normal.

Indeed, if PQ is normal, then $(PQ)^* = QP$ is normal. Since $P(QP) = (PQ)P$, Putnam-Fuglede theorem gives $P^2Q = QP^2$. As P positive, $PQ = QP$. The other way implication is obvious.

Theorem 6 — Let N_1 and N_2 be operators with one of them normal. If $N_1N_2^* = N_2^* N_1$, then each of N_1, N_2 commutes with positive part of the other, but not conversely.

PROOF : Suppose N_1 is normal. Then the relation $N_1N_2^* = N_2^* N_1$ yields by the Fuglede's theorem, $N_1^* N_2^* = N_2^* N_1^*$ so that $N_1N_2 = N_2N_1$. Thus

$$N_1(N_2^* N_2) = (N_1N_2^*) N_2 = N_2^* (N_1N_2) = (N_2^*N_2) N_1.$$

Again $N_2(N_1^* N_1) = (N_2N_1^*) N_1 = N_1^* (N_2N_1) = (N_1^*N_1) N_2$. The converse does not hold as one can see by taking N_1 and N_2 to any two noncommuting unitary operators.

Remarks : (1) Consider for two operators $T_j = U_jP_j (j = 1, 2)$ in their polar decomposition, the following conditions:

c_1 : each (or either) commutes with the adjoint of the other.

c_2 : each commutes with positive part of the other.

c_3 : $U_1P_2 = P_2U_1$, $U_2P_1 = P_1U_2$ and $P_1P_2 = P_2P_1$.

If T_1 and T_2 are two normal operators then $c_1 \Rightarrow c_2$ and $c_2 \not\Rightarrow c_1$ by Theorem 6. Also $c_3 \Rightarrow c_2$ by the proof of Corollary 5. Further, c_1 and c_3 are independent. For, take $T_j = U_jP_j (j = 1, 2)$ where U_j and $P_j (j = 1, 2)$ are operators on $l^2(Z)$, the space of two-way square summable complex sequences with the usual orthonormal basis $\{e_n\}_{n=-\infty}^{\infty}$ defined by $U_1e_1 = e_2$, $U_1e_2 = e_1$ and $U_1e_i = e_i$, $i \neq 1, 2$; $U_2e_{-1} = e_{-2}$, $U_2e_{-2} = e_{-1}$ and $U_2e_i = e_i$, $i \neq -1, -2$; and P_1 and P_2 are projections on $\bigvee_{-\infty}^{-1} \{e_i\}$ and

$\overset{\infty}{\underset{1}{V}}$ $\{e_i\}$ respectively. Here c_1 is satisfied but P_1 does not commute with U_2 so that $c_1 \not\Rightarrow c_3$. Since $c_1 \Rightarrow c_2$ so T_1 and T_2 satisfy c_2 but not c_3. For $c_3 \not\Rightarrow c_1$, take two noncommuting unitary operators.

(2) An operator T is called quasinormal if T commutes with T^*T. If c_2 is satisfied for two quasinormal operators A and B then it is easy to see that products AB and BA are quasinormal. (Arun Bala (1977) has proved this result under stronger assumptions that $AB = BA$, $BA^* = A^*B$.) Using this result and induction on n one can see that A^n $(n \geqslant 0)$ is quasinormal, whenever A is quasinormal.

(3) Note that for quasinormal operators A and B; c_2 and c_1 are independent. $c_2 \not\Rightarrow c_1$ follows from Remark (1), since every normal operator is quasinormal. For $c_1 \not\Rightarrow c_2$ take operators V_1 and V_2 on $l^2(Z)$ defined by

$$V_1 e_i = \begin{cases} e_{i-1} & \text{if } i \leqslant 1 \\ 0 & \text{if } i > 1 \end{cases}$$

$$V_2 e_i = \begin{cases} e_{i+1} & \text{if } i \leqslant 1 \\ 0 & \text{if } i > 1 \end{cases}$$

Then V_1 and V_2 are quasinormal and $V_1 V_2^* = V_2^* V_1$ but

$$V_1(V_2^* V_2)\, e_1 \neq (V_2^* V_2)\, V_1 e_1.$$

In Theorem 3 we had assumed that T_1^* and T_2 satisfy c_2. Here we note that if T_1^* and T_2 satisfy c_2 for two quasinormal operators T_1 and T_2, then $T_1 T_2$ may not be quasinormal. To see this, define

$$T_1 e_i = \begin{cases} e_i + 1 & \text{if } i \geqslant 0 \\ 0 & \text{if } i < 0 \end{cases}$$

and

$$T_2 e_{-1} = \cos\theta e_{-1} + \sin\theta e_0$$

$$T_2 e_0 = -\sin\theta e_{-1} + \cos\theta e_0, \quad 0 < \theta < \pi/2$$

$T_2 e_i = e_i (i \neq -1, 0)$ on $l^2(Z)$. Then T_1 and T_2 are quasinormal operators such that T_1 commutes with $T_2^* T_2$ and T_2 commutes with $T_1 T_1^*$ but $A = T_1 T_2$ is not quasinormal, since $A^*AAe_0 \neq AA^*Ae_0$. (We note that A is hyponormal).

We close this note with an observation about the class θ of Campbell (1975). An operator T is in θ if T^*T commutes with $T + T^*$. For a hyponormal operator

T, if T^* is hyponormal, then T is normal. Here is a similar result for an operator in class θ.

Theorem 7 — If T and T^* are in θ, then T is normal.

PROOF : Since the hypothesis implies that the real part of T commutes with $(T^*T)^{1/2}$ and $(TT^*)^{1/2}$, the result follows from the following result of Embry (1966): An operator A is normal if and only if real part of A commutes with $(A^*A)^{1/2}$ and $(AA^*)^{1/2}$.

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