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ON HERMITIAN OPERATORS ON BANACH SPACES

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This note is concerned with some results on Hermitian operators on Banach spaces.

1. INTRODUCTION

Here (x, y) represents semi-inner product as introduced by Lumer (1961) and Giles (1967).

Giles (1967, Theorem 1) has shown that every normed vector space can be represented as a s.i.p. space.

Definition 1.1 — Let X be a Banach space and $(\, , \,)$ be any consistent s.i.p. on X. Then numerical range of an operator T on X is defined by

$$W(T) = \{(Tx, x) : \| x \| = 1\}.$$

Vidav (1956) defined the following notion of Hermiticity:

Definition 1.2 — An operator T on a Banach space is said to be Hermitian iff $\| I + i\alpha T \| = 1 + o(\alpha)$ for $\alpha \to 0$.

Lumer (1961, Lemma 12) has shown that an operator T is Hermitian iff $W(T)$ is real.

Nieminen (1962) has proved, for the case of Hilbert space, that the operator T is Hermitian iff the following conditions are satisfied:

1. $\sigma(T)$ is real;
2. $\| R_{i\mu} \| \leqslant | \mu |^{-1}$.

where μ is real and $R_{i\mu} = (T - i\mu I)^{-1}$.

In this paper, we generalise Nieminen's result to Banach spaces by semi-inner product techniques.

Here we quote Proposition 2 of Sinclair (1971) which will be needed in the sequel.

Proposition (Sinclair 1971) — Let A be a Banach algebra with identity e. Then

$$\| h + \beta e \| = | \sigma(h + \beta e) |$$

for each Hermitian element h and each complex number β.

2. Hermitian Operators

Theorem 2.1 — Let X be a Banach space. An operator T on X is Hermitian iff $\| R_{i\mu} \| \leqslant | \mu |^{-1}$, μ real.

Proof : Let T be Hermitian. Then $W(T)$ is real. Therefore by Theorem 4 of Lumer (1961), $\sigma(T)$ is real.

Let $\lambda = i\mu$ where μ is any non-zero real number and $i = \sqrt{-1}$.

Now suppose $x \in X$ and $\| x \| = 1$.

Then $| (\{T - \lambda I\} x, x) | = | (Tx, x) - \lambda | \geqslant | \mu |$, as $W(T)$ is real.

or, $$| \mu | \leqslant | (\{T - \lambda I\} x, x) | \leqslant \| (T - \lambda I) x \|.$$

Since $\sigma(T)$ is real, $\| (T - \lambda I)^{-1} \| \leqslant | \mu |^{-1}$.

Thus, $$\| R_{i\mu} \| \leqslant | \mu |^{-1}.$$

Converse : Let $\| R_{i\mu} \| \leqslant | \mu |^{-1}$ for μ real.

If α is a real number then $(I + i\alpha T)(I - i\alpha T) = I + \alpha^2 T^2$.

Thus for $\alpha \to 0$,

$$\begin{aligned} \| I + i\alpha T \| &\leqslant \| (I - i\alpha T)^{-1} \| (1 + o(\alpha)) \\ &= | \alpha |^{-1} R_{-i\alpha} - 1 (1 + o(\alpha)) \\ &\leqslant | \alpha |^{-1} | \alpha | (1 + o(\alpha)) \\ &= 1 + o(\alpha). \end{aligned}$$

Therefore by definition, T is Hermitian.

This completes the result.

Corollary 2.2 — If T is Hermitian, then

$$\| R_\lambda \| \leqslant | \mu |^{-1}$$

where μ is the imaginary part of $\lambda \in C$.

Remark 2.3 : Istratescu (1975, Theorem 1) proved that the operator T is Hermitian iff

$$\| R_{i\mu} \| \leqslant | \mu |^{-1}(1 + o(| \mu |^{-1})), \quad \mu \text{ real}.$$

Remark 2.4 : It would be interesting to know whether every Hermitian operator T satisfies the growth condition G_1,

$$\| R_\alpha \| \leqslant (d(\alpha, \sigma(T)))^{-1} \text{ for } \alpha \notin \sigma(T).$$

Remark 2.5 : It would be interesting to know whether every Hermitian operator is a generalised scalar operator [for literature see Colojoara and Foias (1968)].

Stampfli (1965) has proved that the resolvent of any operator T on a Hilbert space has a first order rate of growth with respect to the closure of numerical range.

We generalise this result to Banach spaces.

Theorem 2.6 — If T is an operator on a smooth Banach space X then T has a first order rate of growth with respect to the closure of numerical range.

PROOF : Since X is smooth, semi-inner product (,) on X is unique. Since T is bounded, $\overline{W(T)}$ is compact. Therefore $\overline{W(T)}$ is not the whole complex plane. Let $\lambda \notin \overline{W(T)}$. Then

$$d = \text{Inf} \{ | y - \lambda | : y \in \overline{W(T)} \} > 0. \qquad \text{...(1)}$$

Now let $z \in X$ and $\| z \| = 1$.

Then $| (\{T - \lambda I\} z, z) | = | (Tz, z) - \lambda | \geqslant d$ by (1)

or, $d \leqslant | (\{T - \lambda I\} z, z) | \leqslant \| \{T - \lambda I\} z \|$

or, $d \leqslant \| (T - \lambda I) z \|$. ...(2)

Again by Theorem 4 of Bonsall and Duncan (1973, p. 22),

$$\sigma(T) \subseteq \overline{W(T)}.$$

Therefore from (2),

$$\| (T - \lambda I)^{-1} \| \leqslant (d(\lambda, \overline{W(T)})^{-1} \text{ for } \lambda \notin \overline{W(T)}$$

which is our required result.

Corollary 2.7 — If $| W(T) | < 1$ then

$$\| (I - \lambda T)^{-1} \| \leqslant (1 - | W(T) |)^{-1} \text{ for } | \lambda | \leqslant 1.$$

PROOF : Let $| \lambda | \leqslant 1$ and $\lambda \neq 0$.

Then $1 - | W(T) | \leqslant 1 - | y | . | \lambda |$ for $y \in \overline{W(T)}$

$\leqslant | 1 - y\lambda |$

or, $1 - |W(T)| \leqslant \operatorname*{Inf}_{y \in \overline{W(T)}} \{ |1 - y\lambda| \} = |\lambda| \,.\, d(\lambda^{-1}, \overline{W(T)}).$

Therefore, by Theorem 2.6,

$$\| (I - \lambda T)^{-1} \| \leqslant (1 - |W(T)|)^{-1}.$$

This completes the proof.

Theorem 2.8 — If P is a polynomial with complex co-efficients whose roots lie on the imaginary axis and T is a Hermitian operator on a Banach space X then

$$|W(P(T))| = |P(|W(T)|)|.$$

PROOF: Since T is Hermitian then $W(T)$ is real and by putting $\beta = 0$ in Proposition 2 of Sinclair (1971),

$$\| T \| = |\sigma(T)| = |W(T)|.$$

Therefore, $|W(T - \alpha I)| = |\, \| T \| - \alpha \,|$ for all imaginary α.

Hence by Proposition 2 of Sinclair (1971),

$$\| T - \alpha I \| = |\, \| T \| - \alpha \,|. \qquad \ldots\text{(i)}$$

Let $P(T) = (T - \alpha_1)(T - \alpha_2) \ldots (T - \alpha_n)$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are not necessarily distinct.

or,
$$\begin{aligned} \| P(T) \| &\leqslant \| T - \alpha_1 \| \ldots \| T - \alpha_n \| \\ &= |\, \| T \| - \alpha_1 | \ldots |\, \| T \| - \alpha_n | \text{ by (i)} \\ &= | P(\| T \|) | = | P(- \| T \|) | \end{aligned}$$

or, $\| P(T) \| \leqslant | P(\| T \|) | = | P(- \| T \|) |.$ ...(ii)

From (ii),

$$\begin{aligned} | P(\| T \|) | &\leqslant \operatorname{Sup} | P(\lambda) | \text{ for } \lambda \in \sigma(T) \\ &\leqslant | W(P(T)) | \leqslant \| P(T) \| \leqslant | P(\| T \|) |. \end{aligned}$$

Therefore, $|W(P(T))| = |P(|W(T)|)|$

which is our required result.

Corollary 2.9 — $|W(P(T))| = \| P(T) \| = |\sigma(P(T))|.$

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