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ON THE GROWTH OF AN ANALYTIC FUNCTION REPRESENTED BY DIRICHLET SERIES

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In the present paper some new growth parameters have been introduced to study precisely the growth of functions of infinite order which are represented by Dirichlet series and are analytic in the half plane and the growth of such functions has been studied with the help of these parameters.

1. INTRODUCTION

Consider the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n) \qquad \ldots(1.1)$$

where $0 \leqslant \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$, $\lambda_n \to \infty$ as $n \to \infty$, $s = \sigma + it$(σ, t being real variables), $\{a_n\}_{n=1}^{\infty}$ is a sequence of complex numbers and

$$\lim_{n\to\infty} \sup (n/\lambda_n) = D < \infty. \qquad \ldots(1.2)$$

If the series given by (1.1) converges absolutely in the half plane Re $s < \alpha(-\infty < \alpha < \infty)$ then it is known (Mandelbrojt 1944, p. 166) that the series (1.1) represents an analytic function in Re $s < \alpha$, and since (1.2) is satisfied we have

$$-\alpha = \lim_{n\to\infty} \sup ((\log | a_n |)/\lambda_n).$$

Let D_α denote the class of all functions $f(s)$ of the form (1.1), which are analytic in the half plane Re $s < \alpha$ $(-\infty < \alpha < \infty)$ and satisfy (1.2). For $f \in D_\alpha$, set

$$M(\sigma) \equiv M(\sigma, f) = \max_{-\infty < t < \infty} | f(\sigma + it) | ,$$

$$m(\sigma) \equiv m(\sigma, f) = \max_{n \geqslant 1} \{ | a_n | \exp (\sigma\lambda_n)\}$$

and $$N(\sigma) \equiv N(\sigma, f) = \max \{n : m(\sigma) = | a_n | \exp (\sigma\lambda_n)\}.$$

$M(\sigma)$, $m(\sigma)$ and $N(\sigma)$ are called, respectively, the maximum modulus, the maximum term and the rank of the maximum term of $f(s)$ for Re $s = \sigma$. The elements in the range set of $N(\sigma)$ are called the principal indices of $f(s)$. It is known (Doetsch 1920) that $\log M(\sigma)$ is an increasing convex function of σ for $\sigma < \alpha$.

For a function $f \in D_\alpha$, Krishna Nandan (1973) has defined the order ρ and lower order $\lambda (0 \leqslant \lambda \leqslant \rho \leqslant \infty)$ of $f(s)$ as

$$\lim_{\sigma \to \alpha} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \log M(\sigma)}{-\log (1 - \exp (\sigma - \alpha))} = \begin{matrix} \rho \\ \lambda \end{matrix} .$$

The above growth parameters do not give any specific information about the growth of $f(s)$ if ρ is either zero or infinite. Recently Awasthi and Dixit (1979) have studied the functions of zero order by comparing the growth of $\log \log M(\sigma)$ with $\log \log (1 - \exp (\sigma - \alpha))^{-1}$. In the present paper an attempt has been made to study the growth of an analytic function for which $\rho = \infty$.

Let Δ_0 be the class of all functions β satisfying the following two conditions:

(i) $\beta(x)$ is defined on $[a, \infty)$, $a > 0$, and is positive, strictly increasing, differentiable and tends to ∞ as $x \to \infty$.

(ii) $\dfrac{d\beta(x)}{d \log x} = o(1)$ as $x \to \infty$.

(In particular we can take $\beta(x) = \log_p x$, $p \geqslant 2$, where $\log_1 x = \log x$ and

$$\log_p x = \log (\log_{p-1} x)).$$

For a function $f \in D_\alpha$ and $\beta \in \Delta_0$, set

$$\begin{matrix} \rho(\beta, f) \\ \lambda(\beta, f) \end{matrix} = \lim_{\sigma \to \alpha} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\beta(\log M(\sigma)}{-\log (1 - \exp (\sigma - \alpha))} .$$

Then $\rho(\beta, f)$ and $\lambda(\beta, f)$ will be called, respectively, β-order and lower β-order of $f(s)$. To avoid some trivial cases we shall assume throughout that $M(\sigma) \to \infty$ as $\sigma \to \alpha$. (Note that these growth parameters give precise information about the growth of the function when its order ρ is infinite). A function $f(s) \in D_\alpha$ is said to be of β-regular growth if $\lambda(\beta, f) = \rho(\beta, f)$, $f(s)$ is said to be of β-irregular growth if $\rho(\beta, f) > \lambda(\beta, f)$.

In section 2 coefficient characterization of $\rho(\beta, f)$ is obtained while coefficient characterization of $\lambda(\beta, f)$ is obtained in section 3. A necessary condition for a function $f(s)$ to be of β-regular growth is also found in section 3. A decomposition theorem for a function of β-irregular growth is obtained in section 4.

2. Coefficient Characterization of $\rho(\beta, f)$

Theorem 1 — If $f(s)$ belongs to D_α and has β-order $\rho(\beta, f)$, then,

$$\rho(\beta, f) = \limsup_{n \to \infty} \frac{\beta(\lambda_n)}{\log \lambda_n - \log^+ (\log | a_n | + \alpha\lambda_n)} \qquad ...(2.1)$$

where $\log^+ x = \max(0, \log x)$.

PROOF : Let the limit superior on the right-hand side of (2.1) be denoted by θ. Clearly $0 \leqslant \theta \leqslant \infty$. First let $0 < \theta < \infty$. Then, for $\theta > \epsilon > 0$ there exist a sequence $\{n_k\}$ of natural numbers such that

$$\log | a_{n_k} | > \lambda_{n_k} \exp(-(1/\bar{\theta})\, \beta(\lambda_{n_k})) - \alpha\lambda_{n_k},\ k = 1, 2, 3, \ldots$$

where $\bar{\theta} = \theta - \epsilon$. Using Cauchy's inequality the above inequality gives for all $\sigma(\sigma < \alpha)$ and all $k = 1, 2, 3, \ldots$

$$\log M(\sigma) \geqslant \log | a_{n_k} | + \sigma\lambda_{n_k} > \lambda_{n_k} \exp(-(1/\bar{\theta})\, \beta(\lambda_{n_k})) + (\sigma - \alpha)\lambda_{n_k}. \quad \ldots(2.2)$$

For $k = 1, 2, 3, \ldots$, set $\sigma_k = \alpha - (\frac{1}{2}) \exp(-(1/\bar{\theta})\, \beta(\lambda_{n_k}))$. Putting, in particular, $\sigma = \sigma_k$ in (2.2) we get

$$\log M(\sigma_k) > \tfrac{1}{2} \lambda_{n_k} \exp(-(1/\bar{\theta})\, \beta(\lambda_{n_k}))$$

or

$$\beta((1/(\alpha - \sigma_k)) \log M(\sigma_k)) > -\bar{\theta} \log(2(\alpha - \sigma_k))$$

or

$$\beta(\log M(\sigma_k)) + \log\ (1/(\alpha - \sigma_k))\, \frac{d\beta(x)}{d \log x}\bigg|_{x = x^*(\sigma_k)}$$

$$> -\bar{\theta} \log(2(\alpha - \sigma_k))$$

where $\log M(\sigma_k) < x^*(\sigma_k) < (1/(\alpha - \sigma_k)) \log M(\sigma_k)$. This easily gives, since $\beta \in \Delta_0$, that

$$\rho(\beta, f) \geqslant \theta \quad \ldots(2.3)$$

(2.3) is obvious for $\theta = 0$. For $\theta = \infty$, the above arguments with an arbitrarily large number in plaee of $\bar{\theta}$ give $\rho(\beta, f) = \infty$.

To prove the reverse inequality assume that $\theta < \infty$, since there is nothing to prove if $\theta = \infty$. Then, given $\epsilon > 0$ and for all $n > n_0 = n_0(\epsilon)$ we have

$$\log | a_n | < \lambda_n \exp(-(1/\theta')\, \beta(\lambda_n)) - \alpha\lambda_n,\ \theta' = \theta + \epsilon.$$

For every $\sigma(\sigma < \alpha)$ we define $n(\sigma)$ as

$$n(\sigma) \leqslant (D + \epsilon')\, \beta^{-1}(-\theta' \log((\tfrac{1}{2})(\alpha - \sigma))) < n(\sigma) + 1 = \bar{n}(\sigma)$$

here $\epsilon' > 0$ is a fixed constant. If σ is sufficiently close to α, using (1.2) we have

$$\sum_{n=n(\sigma)+1}^{\infty} \exp\left(\lambda_n \exp\left(-(1/\theta')\,\beta(\lambda_n)\right) + (\sigma - \alpha)\,\lambda_n\right)$$

$$< \sum_{n=n(\sigma)+1}^{\infty} \exp\left(\lambda_n(\sigma - \alpha)/2\right) < \sum_{n=n(\sigma)+1}^{\infty} \exp\left(\frac{(\sigma - \alpha)n}{2(D + \epsilon')}\right)$$

$$= \frac{\exp\left((\sigma - \alpha)\,(n(\sigma) + 1)/(2(D + \epsilon'))\right)}{1 - \exp\left((\sigma - \alpha)/(2(D + \epsilon'))\right)} = A(n(\sigma)) \text{ (say).}$$

Now,

$$\log A(n(\sigma)) = \frac{(\sigma - \alpha)\,(n(\sigma) + 1)}{2(D + \epsilon')} + \log \frac{2}{(\alpha - \sigma)} + O(1)$$

$$< -\frac{1}{x(\sigma)}\,\beta^{-1}(\theta' \log x(\sigma)) + \log x(\sigma) + O(1)$$

where $x(\sigma) = 2/(\alpha - \sigma)$. Clearly $x(\sigma) \to \infty$ as $\sigma \to \alpha$. Since $\beta \in \Delta_0$ it follows that $\beta^{-1}(\theta' \log x(\sigma)) > (x(\sigma))^2$ for all σ sufficiently close to α. This shows that

$$\log A(n(\sigma)) \to -\infty \text{ as } \sigma \to \alpha$$

i.e., $$A(n(\sigma)) \to 0 \text{ as } \sigma \to \alpha.$$

Consider the function $F(x) = \exp\{x \exp(-\beta(x)/\theta') - (\alpha - \sigma)x\}$. Taking the logarithmic derivative of $F(x)$ and setting it equal to zero we get

$$\frac{F'(x)}{F(x)} = \exp\left(-(\beta(x)/\theta')\right)\left\{1 - (1/\theta')\,\frac{d\beta(x)}{d \log x}\right\} - (\alpha - \sigma) = 0.$$

Since $d\beta(x)/d \log x = o(1)$ as $x \to \infty$ it follows that

$$\tfrac{1}{2} < 1 - \frac{1}{\theta'}\frac{d\beta(x)}{d \log x} < 2 \text{ for } x > x'.$$

Let $\lambda_{n'}$ be a fixed λ_n greater than x' and λ_{n_0}. Then $F'(\lambda_{n'})/F(\lambda_{n'}) > 0$ for $\sigma > \sigma_0$. Also $F'(\lambda_{\bar{n}(\sigma)})/F(\lambda_{\bar{n}(\sigma)}) < 0$ for all $\sigma > \sigma^0$. Now, for $\sigma > \max(\sigma_0, \sigma^0)$ we denote by $x_*(\sigma)$ the point where $F(x_*(\sigma)) = \max\limits_{\lambda_{n'} \leqslant x \leqslant \lambda_{\bar{n}(\sigma)}} F(x)$, then

$$\lambda_{n'} < x_*(\sigma) < \lambda_{\bar{n}(\sigma)} \text{ and } x_*(\sigma) = \beta^{-1}\left(-\theta' \log\left((\alpha - \sigma)/(1 - d(\sigma))\right)\right)$$

where $$d(\sigma) = \frac{1}{\theta'}\left.\frac{d\beta(x)}{d \log x}\right|_{x = x_*(\sigma)}$$

and so $$\max_{n' \leqslant n \leqslant \bar{n}(\sigma)} |a_n| \exp(\sigma\lambda_n) \leqslant \max_{\lambda_{n'} \leqslant x \leqslant \lambda_{\bar{n}(\sigma)}} F(x)$$

(*equation continued on p.* 1228)

$$\leqslant \exp\left\{(\alpha - \sigma)\frac{d(\sigma)}{1-d(\sigma)}\beta^{-1}\left(-\theta' \log\left((\alpha-\sigma)/(1-d(\sigma))\right)\right)\right\}$$

$$\leqslant \exp\{(\alpha - \sigma)\,\beta^{-1}\,(-\theta' \log((\alpha - \sigma)/2))\}. \qquad \ldots(2.5)$$

Now, for $\sigma > \max(\sigma_0, \sigma^0)$ we have

$$M(\sigma) \leqslant \sum_{n=1}^{\infty} |\, a_n \,| \;\exp(\sigma\lambda_n)$$

$$\leqslant P(n') + \Big(\sum_{n=n'+1}^{n(\sigma)} + \sum_{n=n(\sigma)}^{\infty}\Big) \exp\left(\lambda_n \exp(-\beta(\lambda_n)/\theta') + (\sigma - \alpha)\lambda_n\right)$$

where $P(n')$, the sum of first n' terms, is bounded. Using (2.4), (2.5) and the definition of $n(\sigma)$ the above inequality gives

$$M(\sigma) \leqslant P(n') + (D + \epsilon')\,\beta^{-1}\,(-\theta' \log((\alpha - \sigma)/2))$$

$$\times \exp\left\{(\alpha - \sigma)\,\beta^{-1}\left(-\,\theta' \log\frac{(\alpha-\sigma)}{2}\right)\right\} + o(1)$$

or

$$\log M(\sigma) \leqslant \{(\alpha - \sigma)\,\beta^{-1}\,(-\theta' \log((\alpha - \sigma)/2))\}\,(1 + o(1)).$$

Since $\beta \in \Delta_0$, this easily gives

$$\rho(\beta, f) \leqslant \theta. \qquad \ldots(2.6)$$

The theorem now follows from (2.3) and (2.6).

3. Coefficient Characterization of $\lambda(\beta, f)$

We need the following Lemmas. Lemmas 1 and 2 are due to Krishna Nandan (1973).

Lemma 1 — If $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ belongs to D_α, then

$$\log m(\sigma) = \log m(\sigma_0) + \int_{\sigma_0}^{\sigma} \lambda_{N(x)}\, dx, \quad -\infty < \sigma_0 < \sigma < \alpha.$$

Lemma 2 — If $f(s) \in D_\alpha$ satisfies

$$\liminf_{n\to\infty} (\lambda_n - \lambda_{n-1}) = \delta > 0 \qquad \ldots(3.1)$$

then for every $\delta' < \delta$ and for all σ sufficiently close to α,

$$M(\sigma) \leqslant m(\sigma)\left[1 + \frac{1+\delta'}{\delta'}\, N\left(\sigma + \frac{1 - \exp(\sigma - \alpha)}{N(\sigma)}\right)\right]$$

$$\times\; (1 - \exp(\sigma - \alpha))^{-1}.$$

Note: In Lemmas 1 and 2 we assume that $m(\sigma)$ and $N(\sigma)$ are unbounded functions of σ.

Lemma 3 — Let $f(s) \in D_\alpha$. Assume that (3.1) holds and that β-order $\rho(\beta, f)$ of $f(s)$ is non-zero. Then

$$\begin{matrix}\rho(\beta, f) \\ \lambda(\beta, f)\end{matrix} = \lim_{\sigma \to \alpha} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\beta(\log m(\sigma))}{-\log(1 - \exp(\sigma - \alpha))}. \qquad \ldots(3.2)$$

PROOF: Let $\epsilon > 0$ be a fixed number. Since (1.2) is satisfied we have, by Lemma 1, for all σ sufficiently close to α,

$$\begin{aligned} N(\sigma)(\alpha - \sigma) &< (D + \epsilon)\, \lambda_{N(\sigma)}(\alpha - \sigma) \\ &\leqslant 2(D + \epsilon) \int_{\sigma}^{\sigma + \frac{1}{2}(\alpha - \sigma)} \lambda_{N(x)}\, dx \\ &\leqslant 2(D + \epsilon) \log m(\sigma + \tfrac{1}{2}(\alpha - \sigma)). \end{aligned} \qquad \ldots(3.3)$$

Using (3.3), for σ sufficiently close to α, we have

$$\begin{aligned} N(\sigma + \tfrac{1}{2}(1 - \exp(\sigma - \alpha))) &\leqslant 2(D + \epsilon) \\ &\times \frac{\log m(\sigma + \frac{1}{2}(\alpha - \sigma) + \frac{1}{4}(1 - \exp(\sigma - \alpha)))}{(\alpha - \sigma) - \frac{1}{2}(1 - \exp(\sigma - \alpha))} \\ &\leqslant 4(D + \epsilon) \frac{\log m(\sigma + \frac{3}{4}(\alpha - \sigma))}{(\alpha - \sigma)}. \end{aligned} \qquad \ldots(3.4)$$

Now, using Lemma 2 and (3.4) we get

$$\begin{aligned} \log M(\sigma) &\leqslant \log m(\sigma) + \log\log m(\sigma + \tfrac{3}{4}(\alpha - \sigma)) - \log(\alpha - \sigma) \\ &\quad + \log(1 - \exp(\sigma - \alpha))^{-1} + O(1) \\ &\leqslant K \log m(\sigma + \tfrac{3}{4}(\alpha - \sigma)) \log(1 - \exp(\sigma - \alpha))^{-1} \end{aligned} \qquad \ldots(3.5)$$

for all σ sufficiently close to α. Here K is a constant. Now, (3.5) gives

$$\begin{aligned} \beta(\log M(\sigma)) &\leqslant \beta(\log m(\sigma + \tfrac{3}{4}(\alpha - \sigma))) \\ &\quad + \log\log(1 - \exp(\sigma - \alpha))^{-K} \left. \frac{d\beta(x)}{d \log x} \right|_{x = x^*(\sigma)} \end{aligned} \qquad \ldots(3.6)$$

where $\log m(\sigma + \frac{3}{4}(\alpha - \sigma)) < x^*(\sigma) < K \log m(\sigma + \frac{3}{4}(\alpha - \sigma)) \log(1 - \exp(\sigma - \alpha))^{-1}$ This easily gives

$$\rho(\beta, f) \leqslant \limsup_{\sigma \to \alpha} \frac{\beta(\log m(\sigma))}{-\log(1 - \exp(\sigma - \alpha))}. \qquad \ldots(3.7)$$

Next, let the limit inferior on the right-hand side of (3.2) be denoted by φ. To prove $\lambda(\beta, f) \leqslant \varphi$ it is sufficient to consider the case when $\varphi < \infty$. Put $\sigma + \frac{3}{4}(\alpha - \sigma) = \sigma_n$ in (3.6) where $\{\sigma_n\}$ is such that

$$\lim_{n\to\infty} \frac{\beta(\log m(\sigma_n))}{-\log(1-\exp(\sigma_n-\alpha))} = \varphi.$$

Then (3.6) gives

$$\lambda(\beta, f) \leqslant \varphi. \qquad \ldots(3.8)$$

The lemma now follows from (3.7), (3.8) and the fact that $m(\sigma) \leqslant M(\sigma)$ for all $\sigma < \alpha$.

Theorem 2 — Let $f(s) \in D_\alpha$. If $f(s)$ is lower β-order $\lambda(\beta, f)$ then

$$\lambda(\beta, f) \geqslant \liminf_{k\to\infty} \frac{\beta(\lambda_{n_{k-1}})}{\log \lambda_{n_k} - \log^+(\log|a_{n_k}| + \alpha\lambda_{n_k})} \qquad \ldots(3.9)$$

for any increasing sequence $\{n_k\}$ of natural numbers.

PROOF: Let the limit inferior on the right-hand side of (3.9) be denoted by S. First, let $0 < S < \infty$. Then given $\epsilon > 0$, $S > \epsilon$, there exists $k_0 = k_0(\epsilon)$ such that for all $k > k_0$ we have

$$\log|a_{n_k}| > \lambda_{n_k} \exp(-\beta(\lambda_{n_{k-1}})/\bar{S}) - \alpha\lambda_{n_k},\ \bar{S} = S - \epsilon.$$

Choose $\sigma_k = \alpha - \frac{1}{2}\exp(-\beta(\lambda_{n_{k-1}})/\bar{S})$, $k = k_0+1, k_0+2, \ldots$. Let $\sigma_k \leqslant \sigma \leqslant \sigma_{k+1}$, then, using Cauchy's inequality, the above inequality gives

$$\begin{aligned}\log M(\sigma) &\geqslant \log|a_{n_k}| + \sigma\lambda_{n_k} \geqslant \log|a_{n_k}| + \sigma_k\lambda_{n_k}\\ &\geqslant \lambda_{n_k}\exp(-\beta(\lambda_{n_{k-1}})/\bar{S}) - (\alpha-\sigma_k)\lambda_{n_k} = \lambda_{n_k}(\alpha-\sigma_k)\\ &\geqslant (\alpha-\sigma)\,\beta^{-1}(-\bar{S}\log(2(\alpha-\sigma))).\end{aligned}$$

Since $\beta \in \Delta_0$, the above inequality easily gives

$$\lambda(\beta, f) \geqslant S \qquad \ldots(3.10)$$

(3.10) is obvious for $S = 0$. For $S = \infty$, the above analysis, with an arbitrarily large number in place of $\bar{S}$, gives that $\lambda(\beta, f) = \infty$. This proves the theorem.

Corollary — Let $f(s) \in D_\alpha$. Assume that β-order $\rho(\beta, f)$ of $f(s)$ is finite. Further, let (i) $\beta(\lambda_n) \sim \beta(\lambda_{n+1})$ as $n \to \infty$ and

(ii) $$\lim_{n\to\infty} \frac{\beta(\lambda_n)}{\log\lambda_n - \log^+(\log|a_n| + \alpha\lambda_n)} = S_0$$

exists, then $f(s)$ is of β-regular growth and $\lambda(\beta, f) = \rho(\beta, f) = S_0$.

Theorem 3 — Let $f(s) \in D_\alpha$. Assume that β-order $\rho(\beta, f)$ of $f(s)$ is non-zero. Further, let (3.1) be satisfied and let $\psi(n) = (\log|a_n/a_{n+1}|)/(\lambda_{n+1} - \lambda_n)$ be a non-decreasing function of n for $n > n_0$. Then

$$\lambda(\beta, f) = \liminf_{n\to\infty} \frac{\beta(\lambda_{n-1})}{\log \lambda_n - \log^+(\log | a_n | + \alpha\lambda_n)}. \quad ...(3.11)$$

PROOF: As $\psi(n)$ forms a nondecreasing function for $n > n_0$, it follows that $\psi(n) > \psi(n-1)$ for infinitely many values of n, since otherwise $\rho(\beta, f) = 0$. Clearly $\psi(n) \to \alpha$ as $n \to \infty$. When $\psi(n) > \psi(n-1)$, the term $a_n \exp(s\lambda_n)$ becomes the maximum term and we have

$$m(\sigma) = | a_n | \exp(\sigma\lambda_n) \quad \text{for } \psi(n-1) \leqslant \sigma < \psi(n).$$

Now, since (3.1) is satisfied we have by Lemma 3,

$$\lambda(\beta, f) = \liminf_{\sigma\to\alpha} \frac{\beta(\log m(\sigma))}{-\log(1 - \exp(\sigma - \alpha))}.$$

Suppose first that $0 < \lambda(\beta, f) < \infty$. Then given $\epsilon > 0$, there exists $\sigma_1 = \sigma_1(\epsilon)$ such that for $\sigma > \sigma_1$ we have

$$\beta(\log m(\sigma)) \geqslant \bar{\lambda} \log(1 - \exp(\sigma - \alpha))^{-1}$$

here $\bar{\lambda} = \lambda(\beta, f) - \epsilon$. Let $a_{n_1} \exp(s\lambda_{n_1})$ and $a_{n_2} \exp(s\lambda_{n_2})$ $(n_1 > n_0, \psi(n_1 - 1) > \sigma_1)$ be two consecutive maximum terms so that $n_1 \leqslant n_2 - 1$. Then

$$\beta(\log | a_{n_2} | + \lambda_{n_2}\sigma) \geqslant \bar{\lambda} \log(1 - \exp(\sigma - \alpha))^{-1}$$

for all σ satisfying $\psi(n_2 - 1) \leqslant \sigma < \psi(n_2)$, Let $n_1 \leqslant n \leqslant n_2 - 1$. It is easily seen that $\psi(n_1) = \psi(n_1 + 1) = \ldots = \psi(n) = \ldots = \psi(n_2 - 1)$ and that

$$| a_n | \exp(\sigma\lambda_n) = | a_{n_2} | \exp(\sigma\lambda_{n_2}) \quad \text{for } \sigma = \psi(n)$$

Hence,

$$\beta(\log | a_n | + \lambda_n\psi(n)) \geqslant \bar{\lambda} \log(1 - \exp(\psi(n) - \alpha))^{-1}. \quad ...(3.12)$$

Since $\psi(n)$ is nondecreasing, (3.12) gives

$$\beta(\lambda_n) \geqslant \bar{\lambda} \log(1 - \exp(\psi(n) - \alpha))^{-1} \quad ...(3.13)$$

for all sufficiently large values of n.

Again, as $\psi(n)$ forms a nondecreasing function of n for $n > n_0$, we have

$$\log | a_{n+1} | = \log | a_{n_0} | + \sum_{j=n_0}^{n} (\lambda_j - \lambda_{j+1}) \psi(j)$$

$$\geqslant \log | a_{n_0} | + (\lambda_{n_0} - \lambda_{n+1}) \psi(n)$$

therefore

$$\frac{\log \lambda_{n+1} - \log^+(\log | a_{n+1} | + \alpha\lambda_{n+1})}{\beta(\lambda_n)} \leqslant - \frac{\log(\alpha - \psi(n))}{\beta(\lambda_n)} + o(1).$$

Using (3.13), this easily gives that

$$\lambda(\beta, f) \leqslant L \qquad \ldots(3.14)$$

where L is the limit inferior on the right-hand side of (3.11). (3.14) is obvious for $\lambda(\beta, f) = 0$. When $\lambda(\beta, f) = \infty$, the above arguments with an arbitrarily large number in place of $\bar{\lambda}$ give that $L = \infty$.

The theorem now follows in view of (3.14) and Theorem 2.

Theorem 4 — Let $f(s) \in D_\alpha$. Let lower β-order of $f(s)$ be $\lambda(\beta, f)$. Assume that (3.1) holds. Then

$$\lambda(\beta, f) = \max_{\{n_k\}} \left\{ \liminf_{k\to\infty} \frac{\beta(\lambda_{n_{k-1}})}{\log \lambda_{n_k} - \log^+ (\log | a_{n_k} | + \alpha\lambda_{n_k})} \right\} \qquad \ldots(3.15)$$

where maximum in (3.15) is taken over all increasing sequences $\{n_k\}$ of positive integers.

PROOF : First, let β-order $\rho(\beta, f)$ of $f(s)$ be zero. Then $\lambda(\beta, f)$ is also zero. In view of Theorem 2, the maximum given by (3.15) is also zero and so the theorem holds in this case.

Next, let $\rho(\beta, f) > 0$. Now, consider the function $g(s) = \sum_{j=1}^{\infty} a_{n_j} \exp (s\lambda_{n_j})$, where $\{n_j\}_{j=1}^{\infty}$ is the sequence of the principal indices of $f(s)$. It is easily seen that $g(s) \in D_\alpha$ and that $g(s)$ also satisfies (3.1). Further, for any s, $f(s)$ and $g(s)$ have the same maximum term and so, by Lemma 3, β-order and lower β-order of $g(s)$ are the same as those of $f(s)$. Thus $g(s)$ is of lower β-order $\lambda(\beta, f)$. Also,

$$\psi(n_j) = (\log | a_{n_j}/a_{n_{j+1}} |)/(\lambda_{n_{j+1}} - \lambda_{n_j})$$

is a strictly increasing function of j. Since $g(s)$ satisfies the hypothesis of Theorem 3, we have by (3.11)

$$\lambda(\beta, f) = \liminf_{j\to\infty} \frac{\beta(\lambda_{n_{j-1}})}{\log \lambda_{n_j} - \log^+(\log | a_{n_j} | + \alpha\lambda_{n_j})} \, . \qquad \ldots(3.16)$$

But, from Theorem 2, we have

$$\lambda(\beta, f) \geqslant \max_{\{n_k\}} \left\{ \liminf_{k\to\infty} \frac{\beta(\lambda_{n_{k-1}})}{\log \lambda_{n_k} - \log^+ (\log | a_{n_k} | + \alpha\lambda_{n_k})} \right\}. \qquad \ldots(3.17)$$

From (3.16) and (3.17) we get (3.15). This proves the theorem.

Theorem 5 — If $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ belongs to D_α and (3.1) is satisfied, then

$$\lambda(\beta, f) \leqslant \rho(\beta, f) \liminf_{n\to\infty} \frac{\beta(\lambda_{n-1})}{\beta(\lambda_n)} . \qquad \ldots(3.18)$$

PROOF : From (3.15) we get

$$\lambda(\beta, f) \leqslant \max_{\{n_k\}} \left\{ \limsup_{k\to\infty} \frac{\beta(\lambda_{n_k})}{\log \lambda_{n_k} - \log^+ (\log | a_{n_k} | + \alpha\lambda_{n_k})} \right\}$$

$$\times \max_{\{n_k\}} \left\{ \liminf_{k\to\infty} \frac{\beta(\lambda_{n_{k-1}})}{\beta(\lambda_{n_k})} \right\}$$

$$= \left\{ \limsup_{n\to\infty} \frac{\beta(\lambda_n)}{\log \lambda_n - \log^+ (\log | a_n | + \alpha\lambda_n)} \right\}$$

$$\times \left\{ \liminf_{n\to\infty} \frac{\beta(\lambda_n)}{\beta(\lambda_{n+1})} \right\} .$$

Theorem now follows from Theorem 1.

We now give a corollary of the above theorem which shows how the exponents λ_n's influence the growth of a function.

Corollary — If $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ belongs to D_α and (3.1) is satisfied, then

(i) if $f(s)$ is of β-regular growth ($0 < \rho(\beta, f) < \infty$), then $\beta(\lambda_n) \sim \beta(\lambda_{n+1})$ as $n \to \infty$;

(ii) if $\liminf_{n\to\infty} \frac{\beta(\lambda_n)}{\beta(\lambda_{n+1})} = 0$ and $\rho(\beta, f) < \infty$, then $\lambda(\beta, f) = 0$;

(iii) if $\liminf_{n\to\infty} \frac{\beta(\lambda_n)}{\beta(\lambda_{n+1})} = 0$ and $\lambda(\beta, f) > 0$, then $\rho(\beta, f) = \infty$;

(iv) if $\liminf_{n\to\infty} \frac{\beta(\lambda_n)}{\beta(\lambda_{n+1})} < 1$ and $0 < \rho(\beta, f) < \infty$, then $f(s)$ is of β-irregular growth.

4. A DECOMPOSITION THEOREM

Theorem 6 — Let $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ be in D_α. Assume that $f(s)$ is of β-irregular growth and $\lambda(\beta, f) < u < \rho(\beta, f)$. Then $f(s) = g_u(s) + h_u(s)$, where β-order of $g_u(s)$ is less than or equal to u and $h_u(s) = \sum_{p=1}^{\infty} a_{n_p} \exp(s\lambda_{n_p})$ satisfies

$$\lambda(\beta, f) \geqslant u \liminf_{p\to\infty} \frac{\beta(\lambda_{n_{p-1}})}{\beta(\lambda_{n_p})}.$$

PROOF: Let $g_u(s) = \Sigma' a_n \exp(s\lambda_n)$, where Σ' denotes the summation over n for which

$$\log | a_n | \leqslant \lambda_n \exp(-\beta(\lambda_n)/u) - \alpha\lambda_n.$$

Then $g_u(s)$ is in D_α and, by Theorem 1, is of β-order less than or equal to u. Set $h_u(s) = f(s) - g_u(s) = \sum_{p=1}^{\infty} a_{n_p} \exp(s\lambda_{n_p})$, then

$$\log | a_{n_p} | > \lambda_{n_p} \exp(-\beta(\lambda_{n_p})/u) - \alpha\lambda_{n_p}.$$

Let $\sigma_p = \alpha - (\frac{1}{2}) \exp(-\beta(\lambda_{n_p})/u)$. Then, for $\sigma_p \leqslant \sigma \leqslant \sigma_{p+1}$, using Cauchy's inequality we get

$$\begin{aligned}\log M(\sigma) &\geqslant \log | a_{n_p} | + \sigma\lambda_{n_p} \geqslant \log | a_{n_p} | + \sigma_p\lambda_{n_p} \\ &\geqslant \lambda_{n_p} \exp(-\beta(\lambda_{n_p})/u) - (\alpha - \sigma_p)\lambda_{n_p} = \lambda_{n_p}(\alpha - \sigma_p) \\ &\geqslant (\alpha - \sigma)\lambda_{n_p}\end{aligned}$$

or

$$\frac{\beta(\log M(\sigma)) - \log(\alpha - \sigma) \left.\dfrac{d\beta(x)}{d\log x}\right|_{x=x^*(\sigma)}}{-\log(1 - \exp(\sigma - \alpha))} \geqslant \frac{\beta(\lambda_{n_p})}{-\log(1 - \exp(\sigma_{p+1} - \alpha))}$$

where $\log M(\sigma) < x^*(\sigma) < (\log M(\sigma))/(\alpha - \sigma)$. Now, since

$$-\log(1 - \exp(\sigma_{p+1} - \alpha)) \sim \beta(\lambda_{n_{p+1}})/u \text{ as } p \to \infty,$$

the theorem follows from the above inequality.

Added in the Proof: The author has recently come to know that for the particular case $\beta(x) = \log_p x$, $p \geqslant 2$, Dr Krishna Nandan has obtained characterization of $\rho(\beta, f)$ and $\lambda(\beta, f)$ under more restrictive conditions on the exponents λ_n's.

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