Indian J. pure appl. Math., **12**(10) : 1235–1239, October 1981

ON A GENERALIZED INTEGRAL TRANSFORM—II

P. C. MUNOT AND P. A. PADMANABHAM

Department of Mathematics, University of Jodhpur, Jodhpur 342001

(*Received* 31 *January* 1981)

The object of this note, is to establish two theorems, of general character, associated with Hardy's and Srivastava's transforms, and generalized K-transform, defined by the authors earlier (Munot and Padmanabham 1980). In sequel, a result of Prajapat (1975, p. 179) has also been corrected.

1. INTRODUCTION

In a recent paper (Munot and Padmanabham 1980) we have defined the generalized K-transform, viz.

$$K\{f(x_1, \ldots, x_n) : P_1, \ldots, P_n\} = \psi(P_1, \ldots, P_n)$$

$$= \prod_{j=1}^{n} P_j \int_0^\infty \ldots \int_0^\infty k(x_1, \ldots, x_n; P_1, \ldots, P_n)\, f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n \qquad \ldots(1.1)$$

of n-variables, where $f(x_1, \ldots, x_n)$ is a function of n-variables and the kernel of the transform $k(x_1, \ldots, x_n; P_1, \ldots, P_n)$, is so restricted that the integrals on the right remain absolutely convergent.

Most of the integral transforms defined earlier can be easily seen to be special cases of this transform. For example, it reduces to the integral transform defined by Gupta and Mittal (1970), when $x_2 = \ldots = x_n = 0$, and

$$k(x, P) = H^{m,n}_{p,q}\left[xP \,\middle|\, \begin{matrix} (a_s, \alpha_s) \\ (b_s, \beta_s) \end{matrix} \right].$$

2. RESULTS REQUIRED IN THE ANALYSIS

Hardy gave the generalization of the well-known Hankel transform, viz.

$$H_\theta(f(t) : \delta) = \int_0^\infty t J_\theta(\delta t) f(t)\, dt \qquad \ldots(2.1)$$

where $\delta > 0$, and $J_\theta(z)$ is the Bessel function of order θ, in the form,

$$H_\theta^{(\sigma)}(f(t) : \delta) = \int_0^\infty tF_\theta(\delta t)\; f(t)\, dt \qquad \ldots(2.2)$$

where

$$F_\theta(z) = \frac{2^{2-\theta-2\sigma}}{\Gamma(\sigma)\,\Gamma(\theta+\sigma)}\, S_{\theta+2\sigma-1,\theta}(z) \qquad \ldots(2.3)$$

when $\sigma = \theta$, $F_\theta(z) = J_\theta(z)$, the result (2.2) reduces to (2.1). Cooke (1926, p. 381) has obtained the inversion formula, of the Hardy's transform, in the form,

$$f(t) = [H_\theta^{(\sigma)}]^{-1}\,\{H_\theta^{(\sigma)}\,(f(t):\delta)\}$$

$$= \int_0^\infty \delta G_\theta(\delta t)\, H_\theta^{(\sigma)}\,(f(t):\delta)\, d\delta \qquad \ldots(2.4)$$

where, for convenience,

$$G_\theta(z) = \cos(\sigma\pi)\, J_\theta(z) + \sin(\sigma\pi)\, Y_\sigma(z) \qquad \ldots(2.5)$$

provided

(i) $\mathrm{Re}\,(\sigma+1) > 0$, $\mathrm{Re}\,(\theta+\sigma+1) > 0$, $\mathrm{Re}\,(\theta+2\sigma) < \frac{3}{2}$, $\mathrm{Re}\,(\theta) \leqslant \frac{3}{2}$,

(ii) $t^\alpha f(t) \in L(0, \Delta)$, $\alpha = \min\,(1+\theta+2\sigma, \frac{1}{2})$, $\Delta > 0$,

(iii) $t^{1/2} f(t) \in L(\Delta, \infty)$. ...(2.6)

Srivastava (1968, p. 385) has given the following unification of several generalization of the classical Laplace transform,

$$S_{q,k,m}^{(\rho,\sigma)}\,(f(t):P) = \int_0^\infty (Pt)^{\sigma-(1/2)} e^{-Pqt/2}\, W_{k,m}(\rho P t)\, f(t)\, dt \qquad \ldots(2.7)$$

where $W_{k,m}(z)$ is the Whittakar function and

$$f(t) = \begin{cases} O\,(t^{\delta} e^{\epsilon t}), & \text{for small } t > 0 \\ O\,(t^{c}), & \text{for large } t > 0. \end{cases}$$

3. Theorems

Theorem I — Let

$$\psi(P_1, \ldots, P_n) = K\,\{f(x_1, \ldots, x_n) : P_1, \ldots, P_n\}$$

with $\mathrm{Re}\,(P_1) > 0, \ldots, \mathrm{Re}\,(P_n) > 0$, and the integrals

$$\int_0^w \left| \delta^{1\pm\theta} h(\theta)\, \psi(P_1\delta, \ldots, P_n\delta) \right| d\delta \qquad \ldots(3.1)$$

and

$$\int_0^w \left| \delta^{1/2} h(\theta)\, \psi(P_1\delta, \ldots, P_n\delta) \right| d\delta \qquad \ldots(3.2)$$

converge for $w > 0$, then

$$[H_\theta^{(\sigma)}]^{-1}\{h(\delta)\,\psi(P_1\delta,\ldots,P_n\delta)\}$$

$$= \operatorname{cosec}(\theta\pi)\,P_1\ldots P_n\int_0^\infty\ldots\int_0^\infty f(x_1,\ldots,x_n)\,[\sin(\theta+\sigma)\,\pi H_\theta(u(\delta):t)$$

$$-\sin(\sigma\pi)\,H_{-\theta}(u(\delta):t)]\,dx_1\ldots dx_n \qquad \ldots(3.3)$$

provided that the generalized K-transform, of $f(x_1\ldots x_n)$ exists, and the integral remain convergent, along with the conditions (2.6) and where for convenience,

$$u(\delta)=h(\delta)\,\delta^n k(x_1,\ldots,x_n:P_1\delta,\ldots,P_n\delta).$$

PROOF : Making use of the K-transform (1.1) and the Hardy inversion integral (2.4), we have,

$$[H_\theta^{(\sigma)}]^{-1}\{h(\delta)\,\psi(P_1\delta,\ldots,P_n\delta)\}$$

$$=\int_0^\infty \delta G_\theta(\delta t)h(\delta)\,\{(P_1\delta,\ldots,P_n\delta)\int_0^\infty\ldots\int_0^\infty k(x_1,\ldots,x_n:P_1\delta,\ldots,P_n\delta)$$

$$\times f(x_1,\ldots,x_n)\,dx_1\ldots dx_n\}\,d\delta. \qquad \ldots(3.4)$$

Now, changing the order of integration which can be easily justified by virtue of the de la vallēe Poussin's theorem (Bromwich 1931, p. 500), we get

$$[H_\theta^{(\sigma)}]^{-1}\{h(\delta)\,\psi(P_1\delta,\ldots,P_n\delta)\}$$

$$=P_1\ldots P_n\int_0^\infty\ldots\int_0^\infty f(x_1,\ldots,x_n)\,\{\int_0^\infty(\delta)^{n+1}G_\theta(\delta t)\;h(\delta)$$

$$\times\, k(x_1,\ldots,x_n;P_1\delta,\ldots,P_n\delta)\,d\delta\}\,dx_1\ldots dx_n. \qquad \ldots(3.5)$$

Now the theorem would follow when we interpret the δ-integral with the help of (2.4), (2.5) and the result of Erdélyi (1953, p. 9) viz.

$$Y_\theta(z)=\operatorname{cosec}(\theta\pi)\,[\cos(\theta\pi)\,J_\theta(z)-J_{-\theta}(z)]. \qquad \ldots(3.6)$$

Theorem II — If

$$K\{f(x_1,\ldots,x_n):P_1,\ldots,P_n\}=\psi(P_1,\ldots,P_n)$$

with $\mathrm{Re}\,(P_1)>0\ldots\mathrm{Re}\,(P_n)>0$, $\mathrm{Re}\,[(\delta+\theta)\,P'-2\gamma]>0$,

$\mathrm{Re}\,(\sigma+\xi\pm\beta+1)>0$, then

$$S^{(\delta,\sigma)}_{(\theta,\alpha,\beta)}\{g(t)\,\psi(P_1t,\ldots,P_nt):P'\}$$

$$=P_1\ldots P_n\int_0^\infty\ldots\int_0^\infty f(x_1,\ldots,x_n)\,S^{(\delta,\sigma)}_{(\theta,\alpha,\beta)}\{g(t)\,t^n$$

$$\times\, k(x_1,\ldots,x_n:P_1t,\ldots,P_nt):P'\}\,dx_1\ldots dx_n \qquad \ldots(3.7)$$

provided that the generalized K-transform of $|f(x_1, \ldots, x_n)|$ exists, the integrals remain convergent, $g(t) = 0(t^{\xi})$ for small $t > 0$, and $g(t) = 0(t^{\xi}e^{\gamma t})$ for larger $t > 0$.

PROOF : We have from K-transform and (2.7),

$$S^{(\delta,\sigma)}_{(\theta,\alpha,\beta)} \{g(t)\, \psi(P_1t, \ldots, P_nt) : P'\}$$

$$= \int_0^\infty (P't)^{\sigma-(1/2)} e^{-(1/2)\theta P't}\, W_{\alpha,\beta}(\delta P't)\, g(t)\, (P_1t, \ldots, P_nt)$$

$$\times \{\int_0^\infty \ldots \int_0^\infty k(x_1, \ldots, x_n : P_1t, \ldots, P_nt)\, f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n\}\, dt \qquad \ldots(3.8)$$

$$= (P_1 \ldots P_n) \int_0^\infty \ldots \int_0^\infty f(x_1, \ldots, x_n)\, \{\int_0^\infty (P'k)^{\sigma-(1/2)} e^{-(1/2)\theta P't}$$

$$\times W_{\alpha,\beta}(\delta P't)\, g(t)\, t^n k(x_1, \ldots, x_n : P_1t, \ldots, P_nt)\, dt\}\, dx_1 \ldots dx_n. \qquad \ldots(3.9)$$

The inversion of the order of integration is justified since the integrals involved in the process are absolutely convergent. The final result follows when we interpret the t-integral by means of (2.9).

This completes the proof of Theorem II.

4. APPLICATIONS

(i) If we take $n = 2$ in Theorem I, along with the Kernel,

$$K(x, y : P, Q) = H\left[\begin{matrix}(\lambda Px)^{\delta_1} \\ (\mu Qy)^{\delta_2}\end{matrix}\middle|\begin{matrix}(a_s, \alpha_s) \\ (b_s, \beta_s)\end{matrix}\right]$$

it provides us after a little substitution,

$$[H^{(\sigma)}_{\theta}]^{-1} \{h(\delta)\, \phi(f : P\delta, Q\delta)\}$$

$$= \frac{\operatorname{cosec}(\theta\pi)}{\lambda\mu} \int_0^\infty \int_0^\infty f\left(\frac{x}{\lambda P}, \frac{y}{\mu Q}\right) [\sin(\theta + \sigma)\, \pi H_{\theta}(u(\delta) : t)$$

$$- \sin(\sigma\pi)\, H_{-\theta}(u(\delta) : t)]\, dxdy \qquad \ldots(4.1)$$

where for convenience,

$$u(\delta) = h(\delta)\delta^2 H\left[\begin{matrix}(x\delta)^{\delta_1} \\ (y\delta)^{\delta_2}\end{matrix}\middle|\begin{matrix}(a_s, \alpha_s) \\ (b_s, \beta_s)\end{matrix}\right]$$

a result earlier established by Prajapat (1975, p. 179), which had an error of missing the factor δ^2 in expression on right.

(ii) Again taking $x_2 = \ldots = x_n = 0$, and

$$k(x, P) = (xP)^{\alpha} J_{\lambda}^{\mu}(xP)$$

in Theorem I, it yields,

$$[H_{\theta}^{(\sigma)}]^{-1} \{h(\delta)\, \phi(f : P\delta)\}$$

$$= P \operatorname{cosec}(\theta\pi) \int_0^{\infty} f(x) [\sin(\theta + \sigma)\, \pi H_{\theta}(u(\delta) : t)$$

$$- \sin(\sigma\pi)\, H_{-\theta}(u(\delta) : t)]\, dt$$

where $u(\delta)$ stands for

$$h(\delta) \,.\, \delta \,.\, (xP\delta)^{\alpha} J_{\lambda}^{\mu}(xP\delta),$$

a relation between Hardy's transform and Ram Kumar transform defined by Ram Kumar (1954, p. 89).

It is a matter of interest to remark here that, due to the general character of the theorems, proved in section 3, we can fairly easily get many more results by either restricting the number of parameters or by specializing the Kernels therein. For the sake of brevity we are not going into their details.

References

Bromwich, T. J. A. (1931). An Introduction to the Theory of Infinite Series. Macmillan and Co., London.

Cooke, R. G. (1926). The inversion formula of Hardy and Titchmarsh. *Proc. Lond. Math. Soc.* (2), **24**, 381–420.

Erdélyi, A. (1953). Higher Transcendental Functions, Vol. II. McGraw-Hill Book Co., Inc., New York.

Gupta, K. C., and Mittal, P. K. (1970). The *H*-function transform—I. *J. Aust. Math. Soc.*, **11** (2), 142–48.

Hardy, G. H. (1925). Some formulae in the theory of Bessel functions. *Proc. Lond. Math. Soc.* (2), **23**, 61–63.

Munot, P. C., and Padmanabham, P. A. (1980). On a generalized integral transform. *Proc. Nat. Acad. Sci., India*, (communicated).

Prajapat, M. L. (1975). Some problems in the theory of generalized hypergeometric functions, classical polynomials and integral transforms. Ph. D. thesis, Jodhpur University, Jodhpur.

Ram Kumar (1954). Some recurrence relations of the generalized Hankel transform—I. *Ganita*, **5**, 89–94.

Srivastava, H. M. (1968). Certain properties of generalized Whittakar transform. *Mathematica (Cluj)*, **10** (23), 385–90.