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SOME INVARIANCE PROPERTIES OF A SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

R. M. GOEL AND BEANT SINGH MEHROK

Department of Mathematics, Punjabi University, Patiala 147002

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Let $S^*(A, B)$ denote the class of functions $g(z) = z + b_2z^2 + \dots$ regular in the unit disc $E = \{z : |z| < 1\}$ and satisfying the condition

$$\frac{zg'(z)}{g(z)} \prec \frac{1 + Az}{1 + Bz}, \; z \in E, \; -1 \leqslant B < A \leqslant 1.$$

Let $C^*(A, B)$ denote the class of functions $f(z) = z + a_2z^2 + \dots$, regular in E and satisfying the condition $\operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right) > 0$, $z \in E$, where $g \in S^*(A, B)$. In this paper we establish the invariant character of the following operators:

(i) $F_1(z) = \dfrac{(m+1)}{z^m} \displaystyle\int_0^z t^{m-1} f(t)\, dt \quad (m = 1, 2, \dots)$;

(ii) $F_2(z) = \frac{1}{2} \displaystyle\int_0^z \frac{f(t) - f(-t)}{t}\, dt$;

(iii) $f_\lambda(z) = (1 - \lambda)\, z + \lambda f(z) \quad (0 \leqslant \lambda \leqslant 1)$; and

(iv) $f_{\lambda\mu}(z)$ defined by

$$\log f'_{\lambda\mu}(z) = \lambda \log f_1'(z) + \mu \log f_2'(z), \quad \lambda + \mu = 1, \quad \lambda, \mu \geqslant 0,$$

where $f_1, f_2, f \in C^*(A, B)$.

1. INTRODUCTION

Let $\mathcal{P}$ be the class of functions

$$P(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \qquad \dots(1.1)$$

regular in the unit disc $E = \{z : |z| < 1\}$ and satisfying the condition $\operatorname{Re} P(z) > 0$.

Let U be the class of functions

$$w(z) = \sum_{n=1}^{\infty} c_n z^n \qquad \dots(1.2)$$

regular in E and satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$, $z \in E$.

Let S denote the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad \text{...(1.3)}$$

regular and univalent in E.

Let $K(A, B)$ denote the subclass of functions of the form (1.3) and satisfying the condition

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1 + Az}{1 + Bz}, \quad -1 \leqslant B < A \leqslant 1, \quad z \in E. \qquad \text{...(1.4)}$$

Obviously $K(1, -1)$ coincides with K, the class of convex univalent functions in E.

Let $S^*(A, B)$ be the subclass of functions in S which satisfy the condition

$$\frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz}, \quad -1 \leqslant B < A \leqslant 1, \quad z \in E. \qquad \text{...(1.5)}$$

To avoid repitition, we lay down, once for all that $-1 \leqslant B < A \leqslant 1, \quad z \in E$.

$S^*(1, -1)$ is identical with S^*, the class of starlike function in E. Thus $f(z) \in K(A, B)$ if and only if $zf'(z) \in S^*(A, B)$.

Let $C^*(A, B)$ be the subclass of functions in S satisfying the condition

$$\text{Re}\left(\frac{zf'(z)}{g(z)}\right) > 0 \quad (z \in E) \qquad \text{...(1.6)}$$

for some

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S^*(A, B). \qquad \text{...(1.7)}$$

$C^*(1, -1) \equiv C$, the class of close-to-convex functions in E.

By definition of subordination it follows that

$g \in S^*(A, B)$ if and only if

$$\frac{zg'(z)}{g(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}, \quad w(z) \in U. \qquad \text{...(1.8)}$$

Obviously $K(A, B)$ $S^*(A, B)$ and $C^*(A, B)$ are the subclasses of K, S^* and C respectively.

The authors (Goel and Mehrok 1981a) obtained coefficient estimates of functions in $S^*(A, B)$. The class $C^*(A, B)$ was studied earlier by the authors (Goel and Mehrok 1981b) and coefficient estimates, distortion theorems and radius of convexity etc. were obtained.

The purpose of this paper is to establish the invariant character of the following operators, when $f \in C^*(A, B)$.

(i) $F_1(z) = \frac{(m+1)}{z^m} \int_0^z t^{m-1} f(t)\, dt \quad (m = 1, 2, \ldots),$

(ii) $F_2(z) = \frac{1}{2} \int_0^z \frac{f(t) - f(-t)}{t}\, dt,$

(iii) $f_\lambda(z) = (1 - \lambda) z + \lambda f(z) \quad (0 \leqslant \lambda \leqslant 1).$

It is further proved that the set of all points $\log \{f'(z)\}$ for a fixed $z \in E$ and $f(z)$ ranging over the class $C^*(A, B)$ is convex.

2. Preliminary Lemmas

Lemma 1 (Sakaguchi 1959) — If $N(z)$ and $D(z)$ are regular in E, $N(0) = D(0) = 0$; $D(z)$ maps E onto many sheeted region which is starlike with respect to the origin and

$$\operatorname{Re}\left(\frac{N'(z)}{D'(z)}\right) > 0 \text{ in } E, \text{ then}$$

$$\operatorname{Re}\left(\frac{N(z)}{D(z)}\right) > 0.$$

Lemma 2 — The following is a simple consequence of well known Jacks Lemma (Jack 1971).

Let $N(z)$ be regular and $D(z)$ starlike in E and $N(0) = D(0) = 0$. Then for

$$-1 \leqslant B < A \leqslant 1,$$

$$\left| \left(\frac{N'(z)}{D'(z)} - 1\right) \Big/ \left(A - B\frac{N'(z)}{D'(z)}\right) \right| < 1 \text{ implies}$$

$$\left| \left(\frac{N(z)}{D(z)} - 1\right) \right| \Big/ \left| \left(A - B\frac{N(z)}{D(z)}\right) \right| < 1, \quad z \in E. \qquad \ldots(2.1)$$

Lemma 3 — If $g(z) \in S^*(A, B)$, then

$$G_1(z) = \frac{(m+1)}{z^m} \int_0^z t^{m-1} g(t)\, dt \in S^*(A, B) \quad (m = 1, 2, \ldots) \qquad \ldots(2.2)$$

Proof : A simple computation gives

$$\frac{zG_1'(z)}{G_1(z)} = \frac{z^m g(z) - m \int_0^z t^{m-1} g(t)\, dt}{\int_0^z t^{m-1} g(t)\, dt} = \frac{N(z)}{D(z)}, \text{ say.} \qquad \ldots(2.3)$$

It is shown by Bernardi (1969) that

$$D(z) = \int_0^z t^{m-1} g(t)\, dt \text{ is } (m+1) \text{ valent starlike.}$$

Since $\dfrac{N'(z)}{D'(z)} = \dfrac{zg'(z)}{g(z)}$ and $g \in S^*(A, B)$, we have

$$\frac{N'(z)}{D'(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}, \quad w(z) \in U.$$

This is equivalent to

$$\left| \left(\frac{N'(z)}{D'(z)} - 1 \right) \right| \Big/ \left| \left(A - B \frac{N'(z)}{D'(z)} \right) \right| \; | w(z) |$$

which yields, on applying Lemma 2, (2.1) and (2.3),

$$\frac{zG_1'(z)}{G_1(z)} = \frac{N(z)}{D(z)} = \frac{1 + Aw_1(z)}{1 + Bw_1(z)}, \quad w_1(z) \in U,$$

and therefore (2.2) follows.

Lemma 4 — If $g_1(z), g_2(z) \in S^*(A, B)$, then so does

$$g_{\lambda\mu}(z) = g_1(z)^\lambda \, g_2(z)^\mu, \quad \lambda + \mu = 1, \quad \lambda, \mu \geqslant 0.$$

Proof : From the definition of $g_{\lambda\mu}(z)$, we have

$$\frac{zg'_{\lambda\mu}(z)}{g_{\lambda\mu}(z)} = \frac{\lambda z g_1'(z)}{g_1(z)} + \frac{\mu z g_2'(z)}{g_2(z)}. \qquad \ldots(2.4)$$

We have

$$\frac{zg_1'(z)}{g_1(z)} \prec \frac{1 + Az}{1 + Bz}, \quad \frac{zg_2'(z)}{g_2(z)} \prec \frac{1 + Az}{1 + Bz}, \quad z \in E$$

Now $\dfrac{1 + Az}{1 + Bz} \in K$, and hence

$$\lambda \frac{zg_1'(z)}{g_1(z)} + (1 - \lambda) \frac{zg_2'(z)}{g_2(z)} \prec \frac{1 + Az}{1 + Bz} \qquad \text{(Bernardi 1966).} \qquad \ldots(2.5)$$

From (2.4) and (2.5), result follows:

Lemma 5 (Goel and Mehrok 1981b) — If $g \in S^*(A, B)$, then for all

$$|s| \leqslant 1, \; |t| \leqslant 1 (s \neq t),$$

$$\begin{cases} \dfrac{tg(sz)}{sg(tz)} \prec \left(\dfrac{1+Bsz}{1+Btz}\right)^{(A-B)/B}, & B \neq 0, \\ \dfrac{tg(sz)}{sg(tz)} \prec \exp A(s-t)z, & B = 0. \end{cases} \quad \ldots(2.6)$$

Lemma 6 — If $g \in S^*(A, B)$, then

$$\left.\begin{aligned} \left|\arg \frac{g(z)}{z}\right| &\leqslant \frac{(A-B)}{B} \sin^{-1}(Br), \quad B \neq 0 \\ \left|\arg \frac{g(z)}{z}\right| &\leqslant \sin^{-1}(Ar), \quad B = 0. \end{aligned}\right\} \quad \ldots(2.7)$$

PROOF: Consider the case when $B \neq 0$.

Taking $s = 1$, $t = 0$ in (2.6),

$$\frac{g(z)}{z} \prec (1 + Bz)^{(A-B)/B}, \quad B \neq 0$$

and it implies

$$\frac{g(z)}{z} = [1 + Bw_2(z)]^{(A-B)/B}, \quad w_2(z) \in U.$$

Thus for $|z| = r$, we get

$$\left|\arg \frac{g(z)}{z}\right| = \frac{(A-B)}{|B|}\left|\arg(1 + Bw_2(z))\right|$$

$$\leqslant \frac{(A-B)}{|B|} \sin^{-1}(|B| r)$$

$$= \frac{(A-B)}{B} \sin^{-1}(Br).$$

Similarly for $B = 0$, we get from (2.6),

$$\left|\arg \frac{g(z)}{z}\right| \leqslant \sin^{-1}(Ar).$$

The bounds (2.7) are sharp, being attained by the function $g_0(z)$ defined by

$$g_0(z) = \begin{cases} z(1 + B\delta z)^{(A-B)/B}, & B \neq 0 \\ z \exp(A\delta z), & B = 0, \; |\delta| = 1. \end{cases} \quad \ldots(2.8)$$

Corollary — $\text{Re}\left[\frac{g(z)}{z}\right] > 0$ if $\left|\arg \frac{g(z)}{z}\right| < \pi/2$.

That is

$$\frac{(A-B)}{B} \sin^{-1}(Br) \leqslant \pi/2 \text{ or } r \leqslant \frac{1}{B} \sin \frac{B\pi}{2(A-B)}, \; B \neq 0.$$

For $B = 0$, it is obvious that

$$\text{Re} \frac{g(z)}{z} > 0, \quad z \in E.$$

Lemma 7 — If $g(z) \in S^*(A, B)$, then so does

$$G_2(z) = \tfrac{1}{2} \int_0^z \frac{g(t) - g(-t)}{t} \, dt. \qquad \ldots(2.9)$$

PROOF : Let $h(z) \in K(A, B)$. Then

$$1 + \frac{zh''(z)}{h'(z)} = \frac{1 + Aw_3(z)}{1 + Bw_3(z)}, \quad w_3(z) \in U$$

or

$$zh''(z) = w_3(z) \, [Ah'(z) - B(zh'(z))']. \qquad \ldots(2.10)$$

Consider any two points z_1 and z_2 in E. Let L denote the segment $[h(z_1), h(z_2)]$ in the image plane. Since $h(z)$ is convex, it follows that L lies entirely in the image of L, we get

$$\begin{aligned}
&| \{z_2h'(z_2) - z_1h'(z_1)\} - \{h(z_2) - h(z_1)\} | \\
&\qquad = | \int_{L^{-1}} zh''(z) \, dz | \\
&\qquad \leqslant \int_{L^{-1}} | zh''(z) \, dz | \\
&\qquad < \int_{L^{-1}} | Ah'(z) - B(zh'(z))' \, dz | \qquad \text{(using (2.10))} \\
&\qquad = \int_{L^{-1}} | dH(z) | \, , \; H(z) = Ah(z) - B(zh'(z)) \\
&\qquad = H(z_2) - H(z_1) \\
&\qquad = | A(h(z_2) - h(z_1)) - B(z_2h'(z_2) - z_1h'(z_1)) |
\end{aligned}$$

or

$$\left| \frac{z_2h'(z_2) - z_1h'(z_1)}{h(z_2) - h(z_1)} - 1 \right| < \left| A - B \frac{z_2h'(z_2) - z_1h'(z_1)}{h(z_2) - h(z_1)} \right|.$$

Taking $z_2 = z$ and $z_1 = -z$, we have

$$\left| \frac{zh'(z) + zh'(-z)}{h(z) - h(-z)} - 1 \right| < \left| A - B \frac{zh'(z) + zh'(-z)}{h(z) - h(-z)} \right| .$$

It implies

$$\frac{h(z) - h(-z)}{2} \in S^*(A, B). \qquad ...(2.11)$$

On differentiation, (2.9) gives

$$zG_2'(z) = \tfrac{1}{2}[g(z) - g(-z)]. \qquad ...(2.12)$$

$$h(z) \in K(A, B) \Rightarrow$$

$$g(z) = zh'(z) \in S^*(A, B). \qquad ...(2.13)$$

(2.11) can be written as

$$zG_2'(z) = \tfrac{1}{2}[zh'(z) + zh'(-z)]. \qquad ...(2.14)$$

Integrating (2.14) from 0 to z, we see by (2.11),

$$G_2(z) = \tfrac{1}{2}[h(z) - h(-z)] \in S^*(A, B).$$

3. Invariance Properties

Theorem 3.1 — If $f(z) \in C^*(A, B)$, then so does

$$F_1(z) = \frac{(m+1)}{z^m} \int_0^z t^{m-1} f(t)\, dt \quad (m = 1, 2, \ldots). \qquad ...(3.1)$$

Proof : From (2.1) and (3.1), after a little computation, we have

$$\frac{zF_1'(z)}{G_1(z)} = \frac{zf(z) - m \int_0^z t^{m-1} f(t)\, dt}{\int_0^z t^{m-1} g(t)\, dt} = \frac{N_1(z)}{D(z)}, \text{ say.} \qquad ...(3.2)$$

Now

$$D(z) = \int_0^z t^{m-1} g(t)\, dt \text{ is } (m+1) \text{ valent starlike.}$$

Differentiating (3.2), we have

$$\frac{N_1'(z)}{D'(z)} = \frac{zf'(z)}{g(z)} .$$

This gives

$$\operatorname{Re}\left(\frac{N_1'(z)}{D'(z)}\right) = \operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right) > 0.$$

Hence by Lemma 1,

$$\operatorname{Re}\left(\frac{N_1(z)}{D(z)}\right) > 0. \qquad \ldots(3.3)$$

By (3.2) and (3.3), we get

$$\operatorname{Re}\left(\frac{zF_1'(z)}{G_1(z)}\right) > 0, \text{ where } G(z) \in S^*(A, B).$$

Thus

$$F_1(z) \in C^*(A, B).$$

Theorem 3.2 — If $f(z) \in C^*(A, B)$, then so does

$$F_2(z) = \tfrac{1}{2} \int_0^z \frac{f(t) - f(-t)}{t}\, dt. \qquad \ldots(3.4)$$

PROOF : Since $f(z) \in C^*(A, B)$, it follows that there exists $h(z) \in K(A, B)$ such that

$$\operatorname{Re}\left(\frac{f'(z)}{h'(z)}\right) > 0, \; z \in E. \qquad \ldots(3.5)$$

It means there exist positive numbers

$M_1(r)$, $M_2(r)$ ($M_1(r) \geqslant M_2(r)$) such that

$$\left|\frac{f'(z)}{h'(z)} - M_1(r)\right| \leqslant M_2(r), \quad |z| = r < 1. \qquad \ldots(3.6)$$

Proceeding as in Lemma 6, we have

$$\begin{aligned} &|\{f(z_2) - f(z_1)\} - M_1(r)\{h(z_2) - h(z_1)\}| \\ &= \left|\int_{L^{-1}} \{f'(z) - M_1(r)\, h'(z)\}\, dz\right| \\ &\leqslant \int_{L^{-1}} |(f'(z) - M_1(r)\, h'(z))\, dz| \\ &\leqslant \int_{L^{-1}} M_2(r)\, |dh(z)| \qquad \text{(using (3.6))} \\ &= M_2(r)\, |h(z_2) - h(z_1)| \end{aligned}$$

or

$$\left| \frac{f(z_2) - f(z_1)}{h(z_2) - h(z_1)} - M_1(r) \right| \leqslant M_2(r).$$

Changing z_2 to z and z_1 to $-z$,

$$\left| \frac{f(z) - f(-z)}{h(z) - h(-z)} - M_1(r) \right| \leqslant M_2(r). \qquad \ldots(3.7)$$

Consider the transform

$$H_2(z) = \tfrac{1}{2} \int_0^z \frac{h(t) - h(-t)}{t}\, dt. \qquad \ldots(3.8)$$

By Lemma 7, $\frac{1}{2}[h(z) - h(-z)] \in S^*(A, B)$,
and consequently $H_2(z) \in K(A, B)$.

From (3.4) and (3.8), on differentiation, we get

$$\frac{F_2'(z)}{H_2'(z)} = \frac{f(z) - f(-z)}{h(z) - h(-z)}. \qquad \ldots(3.9)$$

(3.7) and (3.9) yield

$$\left| \frac{F_2'(z)}{H_2'(z)} - M_1(r) \right| \leqslant M_2(r).$$

That is,

$$\mathrm{Re}\left(\frac{F_2'(z)}{H_2'(z)}\right) > 0, \quad \text{where } H_2(z) \in K(A, B).$$

Hence $F_2(z) \in C^*(A, B)$.

Remark: The corresponding result for the class C was established by Pommerenke (1965).

Theorem 3.3 — If $f(z) \in C^*(A, B)$, and $f_\lambda(z) = (1 - \lambda) z + \lambda f(z)$, $0 \leqslant \lambda \leqslant 1$. Then

(i) for $|z| < \frac{1}{B} \sin\left(\frac{\pi B}{2(A - B)}\right)$, $B \neq 0$, $f_\lambda(z) \in C^*(A, B)$,

(ii) for $B = 0$, $f_\lambda(z) \in C^*(A, B)$, $z \in E$.

The proof is simple and follows as a direct consequence from the corollary of Lemma 6.

Theorem 3.4 — The set of all points $\log f'(z)$ for a fixed $z \in E$ and $f(z)$ ranging over the class $C^*(A, B)$ is convex.

PROOF: If $f_1(z), f_2(z) \in C^*(A, B)$, then we shall show that the function $f_{\lambda\mu}(z)$ defined by

$$\log f'_{\lambda\mu}(z) = \lambda \log f'_1(z) + \mu \log f'_2(z), \qquad \ldots(3.10)$$

also belongs to $C^*(A, B)$, where $\lambda, \mu \geqslant 0, \lambda + \mu = 1$.

By definition, there exist $g_1(z), g_2(z) \in S^*(A, B)$ such that

$$f'_1(z) = \frac{g_1(z)}{z} P_1(z), \qquad \ldots(3.11)$$

$$f'_2(z) = \frac{g_2(z)}{z} P_2(z), \qquad \ldots(3.12)$$

where $P_1(z), P_2(z) \in \mathcal{P}$.

Using (3.11) and (3.12) in (3.10), we get

$$\log f'_{\lambda\mu}(z) = \log\left[\frac{g_1(z)^\lambda\, g_2(z)^\mu}{z} P_1(z)^\lambda P_2(z)^\mu\right]$$

or

$$f'_{\lambda\mu}(z) = \left[\frac{g_1(z)^\lambda\, g_2(z)^\mu}{z}\right][P_1(z)^\lambda P_2(z)^\mu].$$

By Lemma 4, $g_{\lambda\mu}(z) = g_1(z)^\lambda g_2(z)^\mu \in S^*(A, B)$.

Also

$$P_1(z)^\lambda P_2(z)^\mu = P(z) \text{ for some } P(z) \in \mathcal{P}.$$

Hence the result follows.

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