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ABSOLUTE CESARO SUMMABILITY OF SUCCESSIVELY DERIVED SERIES OF A FOURIER SERIES OF FUNCTIONS OF CLASS $L_p(p > 1)$

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In the present paper the author obtains a criterion for the absolute cesàro summability of the rth derived series of a fourier series of a function of class $L_p(p > 1)$. The case $r = 0$ of the theorem obtained here is equivalent to the theorem proved by Tsuchikura (1954). It is also shown that the order of summability obtained here cannot be improved.

1.1. Let $f(t)$ be a periodic function with period 2π, integrable in the sense of Lebesgue over $(0, 2\pi)$. Let the Fourier series of $f(t)$ be

$$\sum_{1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{1}^{\infty} A_n(t), \text{ say.} \qquad \ldots(1.1.1)$$

The constant term of the Fourier series of $f(t)$ is taken to be zero without any loss of generality.

The rth derived series of the Fourier series of $f(t)$ is given by

$$\sum_{n=1}^{\infty} \left(\frac{d}{dt}\right)^r A_n(t) = \sum_{n=1}^{\infty} A_{n,r}(t) \quad (r \geqslant 0), \text{ say.} \qquad \ldots(1.1.2)$$

When $r = 0$, the series in (1.1.2) reduces to that in (1.1.1).

We write

$$P(t) = \sum_{i=0}^{r-1} \{\theta_i t^i / i!\}$$

where $P(t)$ denotes an arbitrary polynomial subject to the condition that

$$g(t) = [\{f(x+t) - P(t)\} + (-1)^r \{f(x-t) - P(-t)\}]/2t^r \qquad \ldots(1.1.3)$$

satisfies $g(t) \in L(0, 2\pi)$. The function $g(t)$ is defined by periodicity outside the range $(0, 2\pi)$.

We write

$$\phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}$$

1.2. A given infinite series Σa_n is said to be absolutely summable by Cesáro means of order k or summable $|C, k|$ $(k > -1)$ if $\Sigma_n |\tau_n^k - \tau_{n-1}^k| < \infty$, where τ_n^k is the nth Cesáro mean of order k of the sequence $\{na_n\}$ and is given by

$$\tau_n^k = \sum_{s=1}^{n} \{A_{n-s}^{k-1}\, sa_s/A_n^k\} \; (n \geqslant 1)$$

with A_n^k defined by

$$\sum_0^\infty A_n^k x^n = (1-x)^{-k-1} \; (|x| < 1).$$

In this paper, we denote, by $\tau_{n,r}^k(x)$, the nth Cesáro mean of order k $(k \geqslant 0)$ of the sequence $\{nA_{n,r}(x)\}$ and by p' a number such that $p^{-1} + p'^{-1} = 1$.

2.1. Concerning the absolute Cesáro summability of a Fourier series of functions of class $L_p (p > 1)$, Tsuchikura (1954) proved the following theorem :

Theorem A — If $1 < p \leqslant 2$, $\alpha > 1/p'$, and

$$\int_0^\pi \frac{|\varphi(t)|^p}{t} \left| \log \frac{1}{t} \right|^{\alpha p} dt < \infty,$$

then the series (1.1.1) at $t = x$ is summable $|C, k|$, $k > 1/p$.

Our object in this paper is to obtain a criterion for the absolute Cesáro summability of the rth derived series of a Fourier series of a function of class $L_p (p > 1)$. The case $r = 0$ of our theorem is equivalent to Theorem A. We also show that the order of summability obtained in our result cannot be improved. Precisely we prove:

Theorem 1 — Let $1 < p \leqslant 2$, $\alpha > 1/p'$ and

$$\int_0^\pi \frac{|g(t)|^p}{t} \left| \log \frac{\bar{k}}{t} \right|^{\alpha p} dt < \infty \quad (\bar{k} > \pi) \tag{2.1.1}$$

then the series (1.1.2) at $t = x$ is summable $|C, k|$, $k > r + (1/p)$ $(r = 0, 1, 2, \ldots)$.

That the conclusion of our theorem can not be replaced by a sharper one when (2.1.1) holds, by taking in the result $|C, r + (1/p)|$ $(r = 0, 1, \ldots)$ instead of $|C, k|$ for $k > r + (1/p)$, is a consequence of the following theorem :

Theorem 2 — Summability $|C, r + (1/p)|$ $(r = 0, 1, \ldots)$ of the rth derived series of Fourier series of functions of class L_p $(p > 1)$ is not a local property of the generating function.

We shall need the following lemmas for the proof of our theorems :

Lemma 1 — If $0 < \beta \leqslant r+1 \quad (r = 0, 1, \ldots)$, and

$$G_n^{\beta}(t) = \sum_{\nu=0}^{n} A_{n-\nu}^{\beta-1} \sin \nu t$$

then

$$\left(\frac{d}{dt}\right)^{r+1} \{G_n^{\beta}(t)\} = g_n^{\beta}(t) + h_n^{\beta}(t),$$

where

$$h_n^{\beta}(t) = O\,(n^r t^{-\beta-1}) \; (nt \geqslant 1);$$

and

$$g_n^{\beta}(t) = \begin{cases} \dfrac{(i)^{r+1} n^{r+1} \cos \{(n+\frac{1}{2}\beta)\,t - \frac{1}{2}\beta\pi\}}{(2 \sin \frac{1}{2}t)^{\beta}} & (r = 0, 2, 4, \ldots); \\[2ex] \dfrac{(i)^{r+1} n^{r+1} \sin \{(n+\frac{1}{2}\beta)\,t - \frac{1}{2}\beta\pi\}}{(2 \sin \frac{1}{2}t)^{\beta}} & (r = 1, 3, \ldots). \end{cases}$$

Further, uniformly in t,

$$\left(\frac{d}{dt}\right)^{r+1} G_n^{\beta}(t) = O\,(n^{\beta+r+1}).$$

PROOF : The proof involves ideas similar to those used in Zygmund (1959, Vol. I, p. 95 and Vol. II, p. 60). Since with a bit of care one can establish the estimates involved in this lemma, we shall omit its proof.

Lemma 2 — If $1 < p \leqslant 2$, $\alpha > 1/p'$ and

$$\int_0^{\pi} \frac{|\,g(t)\,|^{\,p}}{t} \left|\,\log \frac{\bar{k}}{t}\right|^{\alpha p} dt < \infty \quad (\bar{k} > \pi)$$

then

$$\int_0^{\pi} \frac{|\,g(t)\,|}{t}\; dt < \infty.$$

PROOF : This can be proved with the use of Hölder's inequality.

Lemma 3 (Hyslop 1940) — For $k \geqslant 0$, the series $\Sigma_n (-1)^n n^k$ is summable $|\,C, k+1+\delta\,|$ $(\delta > 0)$.

Lemma 4 (Kogbetliantz 1925) — If Σa_n is summable $|\,C, k\,|$ $(k \geqslant 0)$, then

$$\sum_{n=1}^{\infty} (\mid a_n \mid /n^k) < \infty.$$

3.1. *Proof of Theorem* 1 — Let us suppose that r is even, say $r = 2m$ ($m = 0, 1, \ldots$). Let us assume, as we can, without any loss of generality, that

$$2m + (1/p) < k \leqslant 2m + 1.$$

Now

$$\tau_{n,r}^{k}(x) = \frac{1}{\pi A_n^k} \int_0^{\pi} \{f(x + t) + f(x - t)\} \left(\frac{d}{dt}\right)^{r+1} G_n^k(t)\, dt, \qquad \ldots(3.1.1)$$

where $G_n^k(t)$ is the nth Cesáro mean of order k of the sequence $\{\sin nt\}$.

From (1.1.3) and (4.1.1),

$$\tau_{n,2m}^{k}(x) = \frac{2}{\pi A_n^k} \int_0^{\pi} g(t)\, t^{2m} \left(\frac{d}{dt}\right)^{2m+1} G_n^k(t)\, dt$$

$$+ \frac{1}{\pi A_n^k} \int_0^{\pi} \{P(t) + P(-t)\} \left(\frac{d}{dt}\right)^{2m+1} G_n^k(t)\, dt$$

$$= P + Q, \text{ say.}$$

Thus it will be enough for our purpose to show that

$$\sum_{n=1}^{\infty} (\mid P \mid /n) < \infty \text{ and } \sum_{n=1}^{\infty} (\mid Q \mid /n) < \infty.$$

Now

$$\sum_{n=1}^{\infty} \frac{\mid P \mid}{n} = O\left(\sum_{\nu=1}^{\infty} \sum_{n=2^{\nu}}^{2^{\nu+1}-1} n^{-k-1} \left| \int_0^{\pi} g(t)\, t^{2m} \times \left(\frac{d}{dt}\right)^{2m+1} G_n^k(t)\, dt \right|\right)$$

$$= O\left(\sum_{\nu=1}^{\infty} 2^{-\nu(k+1)} \sum_{n=2^{\nu}}^{2^{\nu+1}-1} \left| \int_0^{\pi/2^{\nu}} g(t)\, t^{2m} \times \left(\frac{d}{dt}\right)^{2m+1} G_n^k(t)\, dt \right|\right)$$

(*equation continued on p.* 1254)

$$+ O\left(\sum_{\nu=1}^{\infty} 2^{-\nu(k+1)} \sum_{n=2^\nu}^{2^{\nu+1}-1} \left| \int_{\pi/2^\nu}^{\pi} g(t)\, t^{2m} \right.\right.$$

$$\left.\left. \times \left(\frac{d}{dt}\right)^{2m+1} G_n^k(t)\, dt \right|\right)$$

$$= \sum\nolimits_1 + \sum\nolimits_2 \quad (say).$$

By Lemma 1,

$$\sum\nolimits_1 = O\left(\sum_{\nu=1}^{\infty} 2^{-\nu(k+1)} \sum_{n=2^\nu}^{2^{\nu+1}-1} n^{k+1+2m} \int_0^{\pi/2^\nu} | (g(t) |\, t^{2m}\, dt)\right.$$

$$= O\left(\sum_{\nu=1}^{\infty} 2^\nu \int_0^{\pi/2^\nu} | g(t) |\, dt\right)$$

$$= O\left(\sum_{\nu=0}^{\infty} 2^\nu \sum_{\rho=\nu}^{\infty} \int_{\pi/2^{\rho+1}}^{\pi/2^\rho} | g(t) |\, dt\right)$$

$$= O\left(\sum_{\rho=0}^{\infty} 2^\rho \int_{\pi/2^{\rho+1}}^{\pi/2^\rho} | g(t) |\, dt\right)$$

$$= O\left(\int_0^{\pi} \frac{| g(t) |}{t}\, dt\right)$$

$$= O(1),$$

by (2.1.1) and Lemma 2.

Again, by Lemma 1,

$$\sum\nolimits_2 \leqslant \sum_{\nu=1}^{\infty} 2^{-\nu(k+1)} \sum_{n=2^\nu}^{2^{\nu+1}-1} \left| \int_{\pi/2^\nu}^{\pi} g(t)\, t^{2m} h_n^k(t)\, dt \right|$$

$$+ \sum_{\nu=1}^{\infty} 2^{-\nu(k+1)} \sum_{n=2^\nu}^{2^{\nu+1}-1} \left| \int_{\pi/2^\nu}^{\pi} g(t)\, t^{2m} g_n^k(t)\, dt \right|$$

$$= \sum\nolimits_{21} + \sum\nolimits_{22}, \text{ say}.$$

Majorizing the inner sum in Σ_{21} by replacing the modulus of the integral by the integral of the modulus of the intergrand and substituting the estimate for $h_n^k(t)$ from Lemma 1, we have

$$\sum_{21} = O\left(\sum_{\nu=1}^{\infty} 2^{-\nu(k+1)} \sum_{n=2^\nu}^{2^{\nu+1}-1} n^{2m} \int_{\pi/2^\nu}^{\pi} | g(t) | \, t^{2m-k-1} \, dt\right)$$

$$= O\left(\sum_{\nu=1}^{\infty} 2^{-\nu(k-2m)} \sum_{\rho=0}^{\nu-1} \int_{\pi/2^{\rho+1}}^{\pi/2^\rho} (| g(t) | / t^{k+1-2m}) \, dt\right)$$

$$= O\left(\sum_{\rho=0}^{\infty} \int_{\pi/2^{\rho+1}}^{\pi/2^\rho} | g(t) | \, t^{-k-1+2m} \, dt \sum_{\nu=\rho+1}^{\infty} 2^{-\nu(k-2m)}\right)$$

$$= O\left(\sum_{\rho=0}^{\infty} 2^{-(\rho+1)(k-2m)} \int_{\pi/2^{\rho+1}}^{\pi/2^\rho} | g(t) | \, t^{-k-1+2m} \, dt\right)$$

$$= O\left(\int_0^\pi \frac{| g(t) |}{t} \, dt\right) = O(1),$$

by (2.1.1) and Lemma 2.

After applying Hölder's inequality to the inner sum of Σ_{22}, we get

$$\sum_{22} \leqslant \sum_{\nu=1}^{\infty} 2^{-\nu(k+1-1/p)} \left\{\sum_{n=2^\nu}^{2^{\nu+1}-1} \left| \int_{\pi/2^\nu}^{\pi} g(t) \, t^{2m} g_n^k(t) \, dt \right|^{p'}\right\}^{1/p'}.$$

By Lemma 1 and Minkowski's inequality

$$\left\{\sum_{n=2^\nu}^{2^{\nu+1}-1} \left| \int_{\pi/2^\nu}^{\pi} g(t) \, t^{2m} g_n^k(t) \, dt \right|^{p'}\right\}^{1/p'}$$

$$= O\left(2^{\nu(1+2m)} \left[\sum_{n=2^\nu}^{2^{\nu+1}-1} \left| \int_{\pi/2^\nu}^{\pi} \frac{g(t) \, t^{2m} \cos\{k(\frac{1}{2}t - \frac{1}{2}\pi) + nt\} \, dt}{(2 \sin t/2)^k} \right|^{p'}\right]^{1/p'}\right)$$

$$= O\left(2^{\nu(1+2m)} \left[\sum_{n=2^\nu}^{2^{\nu+1}-1} \left| \int_{\pi/2^\nu}^{\pi} g(t) \, t^{2m} \frac{\cos k(t-\pi)/2}{(2 \sin t/2)^k} \cos nt \, dt \right|^{p'}\right]^{1/p'}\right)$$

$$+ O\left(2^{\nu(1+2m)} \left[\sum_{n=2^\nu}^{2^{\nu+1}-1} \left| \int_{\pi/2^\nu}^{\pi} g(t) \, t^{2m} \frac{\sin k(t-\pi)/2}{(2 \sin t/2)^k} \sin nt \, dt \right|^{p'}\right]^{1/p'}\right)$$

Writing

$$X(t) = \begin{cases} t^{2m}\, g(t) \, \begin{matrix}\cos \\ \sin\end{matrix} \, (t-\pi)\, \dfrac{k}{2} \Big/ (2 \sin t/2)^k & \left(\dfrac{\pi}{2^\nu} \leqslant t < \pi \right), \\ 0 \quad (0 \leqslant t < \pi/2^\nu); \end{cases}$$

we have by Hausdorff-Young inequality,

$$\sum_{n=2^\nu}^{2^{\nu+1}-1} \left| \int_{\pi/2^\nu}^{\pi} g(t)\, t^{2m}\, g_n^k(t)\, dt \right|^{p'}$$

$$= O\left(2^{\nu(1+2m)} \left\{ \int_0^\pi | X(t) |^p \, dt \right\}^{1/p} \right)$$

$$= O\left(2^{\nu(1+2m)} \left\{ \int_{\pi/2^\nu}^{\pi} \frac{| g(t) |^p}{t^{kp}}\, t^{2mp}\, dt \right\}^{1/p} \right).$$

Hence, by Hölder's inequality,

$$\sum\nolimits_{22} = O\left(\sum_{\nu=1}^{\infty} 2^{-\nu(k-2m-1/p)} \left\{ \int_{\pi/2^\nu}^{\pi} | g(t) |^p \, t^{-(k-2m)p}\, dt \right\}^{1/p} \right)$$

$$= O\left(\left\{ \sum_{\nu=1}^{\infty} \nu^{\alpha p}\, 2^{-\nu(kp-1-2mp)} \int_{\pi/2^\nu}^{\pi} | g(t) |^p \, t^{-(k-2m)p}\, dt \right\}^{1/p} \times \left\{ \sum_{\nu=1}^{\infty} \nu^{-\alpha p'} \right\}^{1/p'} \right)$$

Since, by hypothesis, the series on the right converges when $\alpha > 1/p'$, it will be enough to show that the first factor is bounded. We find

$$\sum_{\nu=1}^{\infty} \nu^{\alpha p}\, 2^{-\nu(kp-1-2mp)} \int_{\pi/2^\nu}^{\pi} | g(t) |^p \, t^{-(k-2m)p}\, dt$$

$$= \sum_{\nu=1}^{\infty} \nu^{\alpha p}\, 2^{-\nu(kp-1-2mp)} \sum_{\rho=\nu-1}^{\infty} \int_{\pi/2^{\rho+1}}^{\pi/2^\rho} | g(t) |^p \, t^{-(k-2m)p}\, dt$$

$$= \sum_{\rho=0}^{\infty} \int_{\pi/2^{\rho+1}}^{\pi/2^\rho} | g(t) |^p \, t^{-(k-2m)p}\, dt \sum_{\nu=0}^{\rho+1} \nu^{\alpha p}\, 2^{-\nu(kp-1-2mp)}$$

(*equation continued on p.* 1257)

$$= O\Big(\sum_{\rho=0}^{\infty} (\rho + 1)^{\alpha p}\, 2^{-(\rho+1)(kp-1-2mp)} \int_{\pi/2^{\rho+1}}^{\pi/2^{\rho}} |\, g(t)\, |^{p}\, t^{-(k-2m)p}\, dt$$

$$= O\Big(\sum_{\rho=0}^{\infty} (\rho + 1)^{\alpha p}\, 2^{(\rho+1)} \int_{\pi/2^{\rho+1}}^{\pi/2^{\rho}} |\, g(t)\, |^{p}\, dt\Big)$$

$$= O\Big(\sum_{\rho=0}^{\infty} (\rho + 1)^{\alpha p}\, 2^{(\rho+1)}\, \pi\, 2^{-\rho} \left|\, \log\left(\frac{\bar{k}}{\pi}\, 2^{(\rho+1)}\right)\right|^{\alpha p} \times \int_{\pi/2^{\rho+1}}^{\pi/2^{\rho}} \frac{|\, g(t)\, |^{p}}{t} \left|\, \log \frac{\bar{k}}{t}\right|^{\alpha p} dt\Big)$$

$$= O\Big(\int_{0}^{\pi} \frac{|\, g(t)\, |^{p}}{t} \left|\, \log \left|\frac{\bar{k}}{t}\right|\right|^{\alpha p} dt\Big) = O(1).$$

The convergence of the series $\Sigma\, (\,|\, Q\,|\, /n)$ is the same as the summability of a series whose general term is

$$\frac{1}{\pi}\int_{0}^{\pi} \{P(t) + P(-t)\} \left(\frac{d}{dt}\right)^{2m} \cos nt\, dt$$

$$= \frac{(-1)^{m}\, n^{2m}}{2\pi} \sum_{i=1}^{m-1} \frac{\theta_{i2}}{(2i)!} \int_{0}^{\pi} t^{2i} \cos nt\, dt$$

$$= (-1)^{m}\, \frac{n^{2m}}{2\pi} \sum_{i=1}^{n-1} \frac{\theta_{2i}}{(2i)!} (-1)^{n} \sum_{s=1}^{i} (-1)^{s-1} \frac{(2i)!}{(2i-2s+1)!} \times n^{-2s}\, \pi^{2i-2s+1}$$

$$= 2\,(-1)^{n} \sum_{s=1}^{m-1} (-1)^{m+s-1}\, n^{2m-2s} \sum_{i=s}^{m-1} \frac{\theta_{2i}\, \pi^{2i-2s}}{(2i-2s+1)!}.$$

By Lemma 3, we find that the series is summable $|\, C, 2m - 1 + \delta\, |\ (\delta > 0)$, and *a fortiori* summable $|\, C, 2m + \delta + (1/p)\, |\ (p > 0)$.

This completes the proof of the theorem when r is even.

The proof when r is odd is similar. On combining both the cases we have the required result.

3.2. *Proof of Theorem* 2 — We first observe that if the Fourier series of a function $f(t)$ is given by $\sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$, then a necessary condition for its rth derived series to be summable $|C, k|$ $(k > 0)$ at $t = 0$, by Lemma 4, is the convergence of

$$\sum_{n=1}^{\infty} |a_n| n^{-k+r} \text{ or } \sum_{n=1}^{\infty} |b_n| n^{-k+r}$$

according as r is even or odd.

It will be sufficient for us to establish the following :

(i) There exists a function of class L_p $(p > 1)$ of period 2π which vanishes in the interval $-\pi/2 \leqslant x \leqslant \pi/2$ such that the $2m$th derived series of the Fourier series of this function is not summable $|C, 2m + (1/p)|$ $(m = 0, 1, \ldots)$, at $x = 0$.

(ii) There exists a function of class $L_p (p > 1)$ of period 2π which vanishes in the interval $-\pi/2 \leqslant x \leqslant \pi/2$, the $(2m + 1)$th derived series of the Fourier series of which is not summable $|C, 2m + 1 + (1/p)|$ $(m = 0, 1, \ldots)$ at $x = 0$.

Before going into the proof of these assertions we like to mention that

$$\sum_{n=2}^{\infty} \frac{\cos nx}{n^{1-(1/p)} \log n} \text{ and } \sum_{n=2}^{\infty} \frac{\sin nx}{n^{1-(1/p)} \log n}$$

are Fourier series of functions of class $L_p (p > 1)$ (see Zygmund 1959, Vol. II, 6.6, Lemma (i), (ii), p. 129).

PROOF OF (*i*) : Let $f(x)$ be given by the following :

$$f(x) = f(-x), f(x + 2\pi) = f(x) \text{ and}$$

$$f(x) = \begin{cases} 0 \quad (0 \leqslant x \leqslant \pi/2) \\ \sum_{n=2}^{\infty} [\cos nx / n^{1-(1/p)} \log n] \quad (\pi/2 < x \leqslant \pi) \end{cases}$$

It is evident that $f(x) \in L_p (p > 1)$. Let a_n be the nth Fourier coefficient of $f(x)$. We have

$$a_{2m} = \frac{2}{\pi} \int_{\pi/2}^{\pi} \cos 2mx \sum_{n=2}^{\infty} \frac{\cos 2nx}{n^{1-(1/p)} \log n} dx$$

$$= (1/2m^{1-(1/p)}) \log m \quad (m = 2, 3, \ldots),$$

the termwise integration being valid.

Now, since,

$$\sum_{m=2}^{\infty} |a_m| m^{-1/p} \geqslant \sum_{m=2}^{\infty} |a_{2m}| (2m)^{-1/p}$$

(*equation continued on p.* 1259)

$$= 2^{-1-(1/p)} \sum_{m=2}^{\infty} (m \log m)^{-1},$$

the proof of (i) is completed by appealing to our observation at the beginning of this section.

PROOF OF *(ii)* : In this case, the function $f(x)$, may, for example, be taken as

$$f(x) = \begin{cases} 0 \quad (0 \leqslant x \leqslant \pi/2), \\ \sum_{n=2}^{\infty} \dfrac{\sin 2nx}{n^{1-(1/p)} \log n} \quad \left(\dfrac{\pi}{2} < x \leqslant \pi\right); \end{cases}$$

$$f(-x) = -f(x), \quad f(x + 2\pi) = f(x).$$

We can follow the reasoning used in the proof of (i) to complete the proof of (ii).

This completes the proof of Theorem 2.

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