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ON THE PROPAGATION OF ELASTIC WAVES IN THE HALF-PLANE

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In a previous investigation (El-Dewik 1975) the problem of propagation of waves in an elastic half-plane has been solved on the assumption that the half-plane is occupied by an ideal elastic medium. It was also assumed that there exists a load perpendicular to the boundary of the half-plane and propagates on the boundary with a constant speed D. The assumption further involved that the displacements are in the direction of the load, whereas the lateral displacements are neglected. However it was mentioned (El-Dewik 1975) that the lateral displacements could be taken into account assuming that they satisfy a similar wave equation to that of the vertical displacement.

The present work is intended as a further contribution in which the lateral displacements are taken into consideration.

The wave equations of the vertical and lateral displacements are given by (see El-Dewik 1975)

$$\frac{\partial^2 W}{\partial t^2} = a^2 \left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} \right) \qquad \ldots(1)$$

$$\frac{\partial^2 V}{\partial t^2} = b^2 \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right) \qquad \ldots(2)$$

where a is the velocity of longitudinal wave and b the velocity of later wave.

These equations will now be solved under the initial conditions:

$$\text{at } t = 0 \quad V = W = 0,\ V_t = W_t = 0 \qquad \ldots(3)$$

and the surface conditions :

$$\left.\begin{aligned} &\delta_{xy} = 0 \ \text{ On the boundary } y = 0 \\ &\delta_{yy} = \delta_0 \ \text{ at } y = 0,\ -Dt \leqslant x \leqslant Dt \\ &\delta_{yy} = 0 \ \text{ at } 0,\ -at < x \leqslant -Dt,\ Dt < x < at. \end{aligned}\right\} \qquad \ldots(4)$$

The solution of the problem may be written in the form:

$$W = \iint_{\sigma_1} \frac{C_1(\xi, \tau)\, d\xi\, d\tau}{[(l_1 - \tau)^2 - ((x - \xi)^2 - y^2)]^{1/2}}, \quad l_1 = at \qquad \ldots(5)$$

$$V = \iint_{\sigma_2} \frac{C_2\,(\xi, \tau)\, d\xi\, d\tau}{[(l_2 - \tau)^2 - (x - \xi)^2 - y^2]^{1/2}}, \quad l_2 = bt \qquad \ldots(6)$$

where

δ_1 = a region in the plane ξ, τ which is limited by hyperploid

$$\tau = l_1 - [(x - \xi)^2 - y^2]^{1/2} \qquad \ldots(7)$$

and the two straight lines

$$\xi = \pm M_1\tau,\ M_1 = \frac{D}{a} \qquad \ldots(8)$$

as shown in Fig. 1.

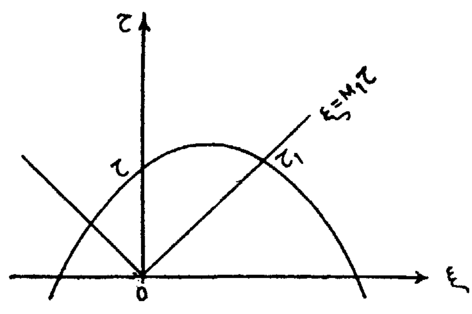

FIG. 1.

δ_2 is similar to δ_1 by exchanging l_1 by l_2 and M_1 by M_2.

It is easy to prove that the expressions (5) and (6) satisfy the wave equations (1) and (2) and the following relations may follow:

$$\left.\begin{aligned} \left(\frac{\partial W}{\partial y}\right)_{y=0} &= -\pi C_1\,(x, l) \\ \left(\frac{\partial V}{\partial y}\right)_{y=0} &= -\pi C_2\,(x, l). \end{aligned}\right\} \qquad \ldots(9)$$

Using the boundary conditions and Hook's law, the relation (9) may be written in the form:

$$(\lambda + 2\mu)\,(-\pi C_1) + \left(\frac{\partial V}{\partial x}\right)_{y=0} = \delta_0 \qquad \ldots(10)$$

$$-\pi C_2 + \left(\frac{\partial W}{\partial x}\right)_{y=0} = 0. \quad \ldots(11)$$

As it is clear from (10) and (11), in order to determine C_1 and C_2 we get two pairs of integral equations. The solution of this set of equations may be reached using the successive iteration.

As a first approximation we put $V = 0$, then from (10) we get

$$C_1(x, l) = -\frac{\delta_0}{(\lambda + 2\mu)}. \quad \ldots(12)$$

Accordingly from (5) an expression for W may be obtained in the form:

$$W = -\frac{\delta_0}{\pi(\lambda + 2\mu)}\int_0^{\tau_0} d\tau \int_0^{M_1\tau} \frac{d\xi}{[(l_1 - \tau)^2 - (x - \xi)^2 - y^2]^{1/2}}$$

$$-\frac{\delta_0}{\pi(\lambda + 2\mu)}\int_0^{\tau_1} d\tau \int_{\xi_1}^{M_1\tau} \frac{d\xi}{[(l_1 - \tau)^2 - (x - \xi)^2 - y^2]^{1/2}} \quad \ldots(13)$$

where

$\xi_1 = x - [(l_1 - \tau)^2 - y^2]^{1/2}$

$\tau_0 =$ may be obtained by solving the set of equations of hyperbolic (7) and the straight line $\xi = 0$ i.e.,

$$\tau_0 = l_1 - [x^2 - y^2]^{1/2} \quad \ldots(14)$$

$\tau_1 =$ may be obtained by solving the set of equations of hyperbolic (7) and the two straight lines

$\xi = \pm M_1\tau$ i.e.,

$$\tau_1 = \frac{1}{1 - M_1^2}\,[-(M_1x - l_1) \pm \{(M_1x - l_1)^2 - (1 - M_1^2)(l_1^2 - x^2 - y^2)\}^{1/2}] \quad \ldots(15)$$

By differentiating eqn. (13) with respect to x and y we get

$$\frac{\partial W}{\partial y} = \frac{\delta_0}{(\lambda + 2\mu)}\left(\sin^{-1}\frac{x}{[(l_1 - \tau_0)^2 - y^2]^{1/2}}\right)\frac{\partial \tau_0}{\partial y}$$

$$-\frac{\delta_0}{(\lambda + 2\mu)}\left(\sin^{-1}\frac{x}{\{(l_1 - \tau_1)^2 - y^2\}^{1/2}}\right)\frac{\partial \tau_1}{\partial y} +$$

(equation continued on p. 1263)

$$+ \frac{\delta_0}{(\lambda + 2)} (x - M_1 l_1 - M_1 y) \int_0^{l_1 - \tau_1} \frac{dz}{(z + y)\sqrt{R_1}}$$

$$+ \frac{\delta_0}{(\lambda + 2\mu)} \int_{l_1}^{l_1 - \tau_1} \frac{dz}{(z - y)\sqrt{R_1}} \qquad \ldots(16)$$

$$\frac{\partial W}{\partial x} = \frac{\delta_0}{(\lambda + 2\mu)} \left(\sin^{-1} \frac{x}{[(l_1 - \tau_0)^2 - y^2)]^{1/2}} \right) \frac{\partial \tau_0}{\partial x}$$

$$- \frac{\delta_0}{(\lambda + 2\mu)} \left(\sin^{-1} \frac{x}{[(l_1 - \tau_1) - y^2]^{1/2}} \right) \frac{\partial \tau_1}{\partial x}$$

$$- \frac{\delta_0}{(\lambda + 2\mu)} \int_{l_1}^{l_1 - \tau_1} \frac{dz}{\sqrt{R_1}} \frac{\delta_0}{(\lambda + 2\mu)} \int_{l_1}^{l_1 - \tau_1} \frac{dz}{[(z^2 - x^2 - y^2)]^{1/2}} \qquad \ldots(17)$$

where

$$R_1 = (1 - M_1^2) z^2 + 2M_1(x - M_1 l_1) z - Y^2 - (x - M_1 l_1)^2.$$

From (11) we have

$$C_2 = \frac{1}{\pi} \left(\frac{\partial W}{\partial x} \right)_{y=0}. \qquad \ldots(18)$$

Accordingly from (6) an expression for $V(x, y, t)$ may be obtained in the form

$$V = \frac{1}{\pi} \int_0^{\bar{\tau}_0} d\tau \int_0^{M_2 \tau} \frac{\left(\frac{\partial W}{\partial x} \right)_{y=0} d\xi}{\{(l_2 - \tau)^2 - (x - \xi)^2 - y^2\}^{1/2}}$$

$$- \frac{1}{\pi} \int_0^{\bar{\tau}_1} d\tau \int_{\xi_1}^{M_2 \tau} \frac{\left(\frac{\partial W}{\partial x} \right)_{y=0} d\xi}{\{(l_2 - \tau)^2 - (x - \xi)^2 - y^2\}^{1/2}} \qquad \ldots(19)$$

where $\bar{\tau}_0$, $\bar{\tau}_1$ and $\bar{\xi}_1$ can be obtained by replacing l_1 and M_1 by l_2 and M_2 in the expressions given for τ_0, τ_1 and ξ_1.

By differentiating eqn. (19) with respect to x and y we get

$$\frac{\partial V}{\partial x} = \frac{1}{\pi} \left(\frac{\partial W}{\partial x} \right)_{y=0} \left(\sin^{-1} \frac{x}{\{(l_2 - \bar{\tau}_0)^2 - y^2\}^{1/2}} \right) \frac{\partial \bar{\tau}_0}{\partial x}$$

$$- \frac{1}{\pi} \left(\frac{\partial W}{\partial x} \right)_{y=0} \left(\sin^{-1} \frac{x}{\{(l_2 - \bar{\tau}_1) - y^2\}^{1/2}} \right) \frac{1}{x} +$$

(*equation continued on p.* 1264)

$$+ \frac{1}{\pi} \frac{\partial}{\partial x}\left(\frac{\partial W}{\partial x}\right)_{y=0} \int_{0}^{\bar{\tau}_0} \sin^{-1} \frac{x - M_2\tau}{\{(l_2 - \tau)^2 - y^2\}^{1/2}}\, d\tau$$

$$- \frac{1}{\pi} \frac{\partial}{\partial x}\left(\frac{\partial W}{\partial x}\right)_{y=0} \int_{0}^{\bar{\tau}_0} \sin^{-1} \frac{x}{\{(l_2 - \tau)^2 - y^2\}^{1/2}}\, d\tau$$

$$+ \frac{1}{\pi}\left(\frac{\partial W}{\partial x}\right)_{y=0} \int_{l_2}^{l_2 - \bar{\tau}_1} \frac{dz}{\sqrt{R_1}} + \frac{1}{\pi}\left(\frac{\partial W}{\partial x}\right)_{y=0}$$

$$\times \int_{l_2}^{l_2 - \bar{\tau}_1} \frac{dz}{\{z^2 - x^2 - y^2\}^{1/2}} \qquad \ldots(20)$$

$$\frac{\partial V}{\partial y} = \frac{1}{\pi}\left(\frac{\partial W}{\partial x}\right)_{y=0} \sin^{-1} \frac{x}{[(l_2 - \tau_0)^2 - y^2]^{1/2}} \frac{\partial \bar{\tau}_0}{\partial y}$$

$$- \frac{1}{\pi}\left(\frac{\partial W}{\partial y}\right)_{y=0} \sin^{-1} \frac{x}{[(l_2 - \tau_1)^2 - y^2]^{1/2}} \frac{\partial \bar{\tau}_1}{\partial y}$$

$$+ \frac{1}{\pi}\left(\frac{\partial W}{\partial x}\right)_{y=0} (x - M_2\, l_2 - M_2 y) \int_{l_2}^{l_2 - \bar{\tau}_1} \frac{dz}{(z + y)\, [R_1]^{1/2}}$$

$$+ \frac{1}{\pi}\left(\frac{\partial W}{\partial x}\right)_{y=0} \int_{l_2}^{l_2 - \bar{\tau}_1} \frac{dz}{(z^2 - x^2 - y^2)^{1/2}}. \qquad \ldots(21)$$

Knowing the components of strains it is easy to find the stresses by use of Hook's law. Putting the value of C_2 in (10) we can find C_1 in the second approximation, then we have:

$$C_1(x, l) = \frac{\delta_0 - \left(\frac{\partial V}{\partial x}\right)_{y=0}}{\pi(\lambda + 2\mu)} = f_1(x, l). \qquad \ldots(22)$$

From (22) and (5) we can get W in the second approximation

$$W = \frac{f_1(x, l)}{\pi(\lambda + 2\mu)} \int_{0}^{\tau_0} d\tau \int_{0}^{M_1\tau} \frac{M_1 d\xi}{[(l_1 - \tau)^2 - (x - \xi)^2 - y^2]^{1/2}}$$

$$- \frac{f_1(x, l)}{\pi(\lambda + 2\mu)} \int_{0}^{\tau_0} d\tau \int_{\xi_1}^{M_1\tau} \frac{d\xi}{[(l_1 - \tau)^2 - (x - \xi)^2 - y^2]^{1/2}}$$

The second approximation of C_2 can now be obtained for eqn. (11) by substituting for C_1 from eqn. (22) i.e.

$$C_2 = -\frac{1}{\lambda + 2\mu}\frac{\partial}{\partial x}\iint_{\delta_2}\frac{\delta_0 - \left(\frac{\partial W}{\partial x}\right)_{y=0} d\xi\, d\tau}{[(l_2 - \tau)^2 - (x - \xi)^2 - y^2]^{1/2}} = f_2(x, l_2).$$

By putting the function $f_2(x, l_2)$ instead of the term $\left(\frac{\partial W}{\partial x}\right)_{y=0}$ in eqns. (19), (20) and (21) we obtain similar expressions for V, $\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial y}$.

The results for $y = 0.5$ and for different values of x are shown in Figs. 2–5.

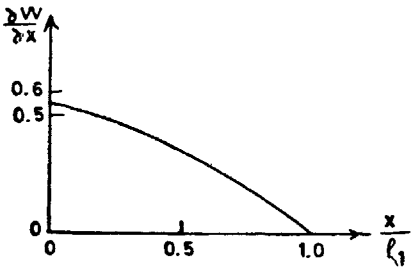

FIG. 2.

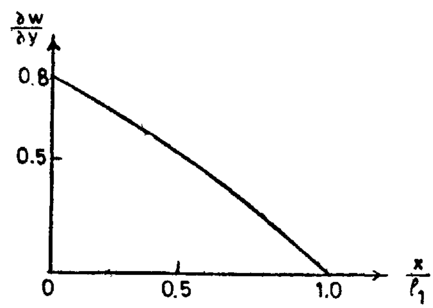

FIG. 3.

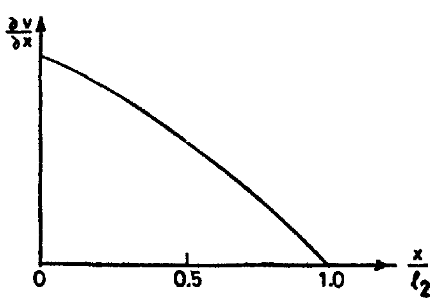

FIG. 4.

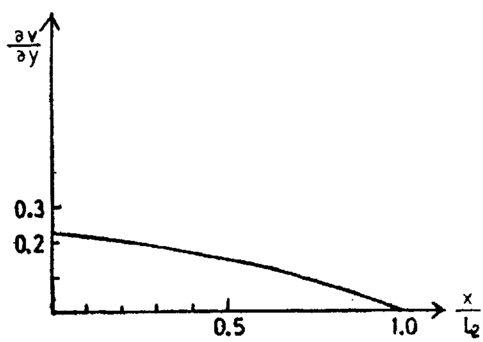

FIG. 5.

REFERENCE

El-Dewik, Fawze Shaban (1975). On the propagation of elastic waves in the half-plane. *Spornik Asperantor Mokh. Institute*, N. 2. (In Russian).