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NEUMANN FUNCTION FOR THE SPHERE—I

B. M. NAYAR

Department of Applied Mathematics, Thapar College of Engineering, Patiala 147001

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Poisson's integral solving the Dirichlet problem or the first boundary value problem for the sphere is too well known, where the classical method of images is used to formulate the appropriate Green's function of the first kind. The integral is interpreted as the sum of potentials due to single and double layers i.e. spreads of monopoles and dipoles, respectively, on the surface.

Herein, analogous results are stated for the Neumann problem or the second boundary value problem. With the help of the method of images appropriate Neumann function or the Green's function of the second kind is constructed and expressed in terms of eigenfunctions associated with the surface. In this case, spherical and ultra spherical harmonics appear.

The surface integral solving the Neumann problem is likewise interpreted as the sum of potentials due to monopoles and 'multipoles', respectively. The converse problem is also established and complete agreement with the Dirichlet problem is exhibited for the interior and exterior regions.

The theory of harmonic functions helps establish uniqueness and certain symmetric properties of the Neumann function. Expansion in terms of fundamental functions is given and convergence of the series established. It is also seen that the potential on the surface of the sphere satisfies a Fredholm integral equation of the second kind having a symmetric and weakly singular kernel.

1. INTRODUCTION

In dealing with a mixed boundary value problem in an earlier paper (Nayar 1975) we referred to the Green's function of the first and the second kind for a smooth closed surface in a general way. For a symmetric surface like a sphere, Green's function of first kind leading to Poisson's integral is too well known. The purpose herein is to treat the adjoint problem.

Let S be the surface of a sphere of radius a with centre O and R, R' be regions interior (negative) and exterior (positive) to it. Let P and $Q \in R$ and $P' \in R'$ be the inverse of P with respect to S.

Let the points referred to the spherical polar system be $P(\rho, \theta, \varphi)$, $P'(a^2/\rho, \theta, \varphi)$ and $Q(\rho', \theta', \varphi')$ if $PQ = r$ and $P'Q = r'$, then

$$r^2 = \rho^2 + \rho'^2 - 2\rho\rho' \cos \gamma \qquad \text{...(1.1)}$$

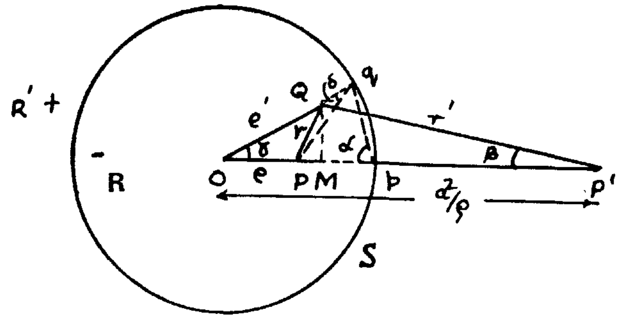

FIG. 1.

$$r'^2 = \left(\frac{a^2}{\rho}\right)^2 + \rho'^2 - (2a^2/\rho)\rho' \cos \gamma \qquad \text{...(1.2)}$$

where

$$\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\varphi - \varphi'). \qquad \text{...(1.3)}$$

Then Green's function of the first kind with pole at P is given by (Kellogg 1929, p. 240)

$$G(Q, P) = \frac{1}{r} - \frac{a}{\rho r'} . \qquad \text{...(1.4)}$$

If $Q \to q$ on S and ν represents the outward directed normal to it, then any function U harmonic in R is given by (Kellogg 1929 p. 241)

$$U(P) = -(1/4\pi)\left(U, \frac{\partial G}{\partial \nu}\right)_S \qquad \text{...(1.5)}$$

where the subscript indicates a surface integral in q over S.

For a sphere of radius a, it becomes the well-known Poisson formula

$$U(P) = \frac{a^2 - \rho^2}{4\pi a} (f, r^{-3})_S \qquad \text{...(1.6)}$$

which solves the Dirichlet problem for the interior, where $f(\theta, \varphi)$ is the pre-assigned continuous value of $U(P)$ on S. For the exterior Dirichlet problem f is replaced by $-f$.

The right-hand side above has physical interpretation since it may be written as

$$U(P) = -(1/4\pi a)(f, r^{-1})_S - (1/2\pi)\left(f, \frac{\partial}{\partial \nu} (1/r)\right)_S \qquad \text{...(1.7)}$$

which is the sum of potentials of a single and a double layer on S with continuous density and moment, respectively.

Ideas similar to the above are employed in dealing with the Neumann problem or the second boundary value problem for the sphere. Neumann function or the Green's function of the second kind is constructed and analogous results stated.

2. Construction of Neumann Function

We recall that a function $V(P)$ harmonic in R, admits a representation in terms of its normal derivative on S (Kellogg 1929, p. 246) as

$$V(P) = \frac{1}{4\pi}\left(N, \frac{\partial V}{\partial \nu}\right)_S - \frac{C}{4\pi}(V, 1)_S \qquad \ldots(2.1)$$

where N denotes the Neumann function much that on S, $\frac{\partial N}{\partial \nu} = C$, a constant. In the particular case of a sphere of radius a, since $C = -1/a^2$, the second term on the right above, by virtue of Gauss' theorem of arithmetic mean (Kellogg 1929, p. 223), represents the value $V(0)$, at the centre of the sphere.

If $g(\theta, \varphi)$ is the prescribed continuous normal derivative of V on S, then

$$\frac{\partial V}{\partial \nu} = g(\theta, \varphi) \qquad \ldots(2.2)$$

and

$$V(P) = \frac{1}{4\pi}(N, g)_S + V(O), \qquad \ldots(2.3)$$

solves the Neumann problem for the interior of the sphere subject to the constraint [Kellogg 1929, p. 212]

$$(1, g)_S = 0. \qquad \ldots(2.4)$$

We wish to construct N, an analytic function, having properties analogous to those of G, i.e.,

(i) N is symmetric and has continuous partial derivatives of the first order in any closed portion of R which does not contain P.

(ii) $\frac{\partial N}{\partial \nu} = -\frac{1}{a^2}$ on S.

(iii) $\nabla^2 N = 0$ in R.

Taking a clue from the classical method of images, we set

$$N(Q, P) = (1/r) + (a/\rho r') + X(a, \rho', \rho, r, r') \qquad \ldots(2.5)$$

where X has somewhat similar properties. It may have the dimensions of inverse distance as is the case with the other two terms above.

Due to radial symmetry, ν coincides with ρ' and $\rho r' = ar$ on S. (1.1), becomes, for $\rho' = a$

$$r^2 = a^2 + \rho^2 - 2a\rho \cos \gamma \qquad \ldots(2.6)$$

which is equivalent to

$$\frac{a - r}{\rho} = \frac{(a + r) \cos \gamma - \rho}{(a + r) - \rho \cos \gamma}. \qquad \ldots(2.7)$$

Let the normal derivatives be denoted with subscripts ρ' and their values on S with $\hat{\rho}'$, we then have

$$\left(\frac{1}{r} + \frac{a}{\rho r'}\right)_{\hat{\rho}'} = -1/ar. \qquad \ldots(2.8)$$

Differentiating (2.5) and using (2.8) and property (ii) of N above, we have

$$X_{\hat{\rho}'} = (a - r)/a^2 r. \qquad \ldots(2.9)$$

Replacing ar by $\rho r'$ wherever it occurs and using (2.7), we get

$$X_{\rho'} = -\frac{1}{a} \frac{(a^2 - \rho\rho' \cos \gamma + \rho r')_{\rho'}}{(a^2 - \rho\rho' \cos \gamma + \rho r')}. \qquad \ldots(2.10)$$

Integrating partially with respect to ρ' and assuming $X = 0$ when $\rho' = 0$, we have

$$X = \frac{1}{a} \log \frac{2a^2}{a^2 - \rho\rho' \cos \gamma + \rho r'}. \qquad \ldots(2.11)$$

The assumption is valid since the potential at the centre O due to a unit charge at P and a charge of amount a/ρ at P' is $(1/\rho) + (1/a)$. This fact is comparable to the property $G = 0$ on S and is symmetric in R.

Evidently N has the dimensions of inverse distance. It has the requisite analytic properties also. Detailed calculations, though tedious, are outlined in the appendix.

Let β be the angle $OP'Q$ and QM, the perpendicular from Q on OP', then we have

$$X = \frac{1}{a} \log [2a^2/\rho r'(1 + \cos \beta)]. \qquad \ldots(2.12)$$

An obscure geometric argument is advanced by Kellogg (1929, p. 247) that the logarithm of a linear combination of distance from a fixed point P' and its projection on the fixed line OPP' is harmonic. This needed verification in the light of the fact that logarithmic potentials are harmonic in two dimensions, so it is pertinent to establish that X is indeed harmonic in three dimensions.

3. Simple System of Co-ordinates

Choosing a simple system of co-ordinates and setting $\theta = \varphi = 0$, without loss of generality, we have $\gamma = \theta'$ and $P(\rho, 0, 0)$, $P'(a^2/\rho, 0, 0)$ and $Q(\rho', \theta', \varphi')$, whence

$$r^2 = \rho^2 - 2\rho\rho' \cos\theta' + \rho'^2 \qquad ...(3.1)$$

$$r'^2 = (a^2/\rho)^2 + \rho'^2 - 2(a^2/\rho)\,\rho' \cos\theta'. \qquad ...(3.2)$$

In polar spherical form then (Kellogg 1929, p. 183)

$$\nabla^2 N = \rho'^2 N_{\rho'\rho'} + 2\rho' N_{\rho'} + N_{\theta'\theta'} + \cot\theta' N_{\theta'}. \qquad ...(3.3)$$

It is easy to show that this vanishes for the first two constituent term of N. For the third term $X(\rho', \theta')$, we have if

$$D_1' = a^2 - \rho\rho' \cos\theta' + \rho r' \qquad ...(3.4)$$

$$\nabla^2 X = \rho'^2\{(a^2 + \rho r')^2 - 2\rho\rho'(a^2 + \rho r')\cos\theta' + \rho^2\rho'^2\} \times (D_1'^2\, ar'^2)^{-1} - \frac{2\rho\rho'^2}{ar' D_1'} \qquad ...(3.5)$$

which vanishes since the terms in the curly brackets simplify to $2\rho r' D_1'$ by virtue of (3.2) and (3.4) which establishes property (iii), of N above. Furthermore, letting $\rho' \to a$, we get

$$N_{\hat{\rho}'} = -\frac{1}{ar^3}(a^2 - 2a\rho\cos\theta' + \rho^2)/ar^3 - \frac{\rho(\rho - a\cos\theta' - r\cos\theta')}{a^2 r(a - \rho\cos\theta' + r)} \qquad ...(3.6)$$

which simplifies by virtue of (2.7) and (3.1) for $\gamma = \theta'$ to

$$N_{\hat{\rho}'} = -\frac{r^2}{ar^3} + \left(\frac{\rho}{a^2 r}\right)\left(\frac{a - r}{\rho}\right) = -\frac{1}{a^2} \qquad ...(3.7)$$

which verifies property (ii) of N. The above calculations become cumbersome for $N(\rho', \theta', \varphi')$ when polar variation is taken into account. Calculations are detailed in the Appendix. From the foregoing we have:

Theorem 1 — The function defined by

$$N(Q, P) = (1/r) + (a/\rho r') + X(Q, P), \qquad ...(3.8)$$

where

$$X(Q, P) = (1/a)\log\{2a^2/(a^2 - \rho\rho'\cos\gamma + \rho r')\} \qquad ...(3.9)$$

satisfies the conditions for being the Neumann function for the sphere of radius a.

4. Solution of the Neumann Problem

Denoting values on S by $\hat{}$, (3.8) and (3.9) give

$$\hat{N} = 2/r + \hat{X}, \qquad ...(4.1)$$

where $$\hat{X} = \frac{1}{a}\log\{2a/(a - \rho\cos\gamma + r)\}. \qquad ...(4.2)$$

Substituting in (2.3), we have

$$V(P) = \frac{1}{2\pi}(g, 1/r)_S + \frac{1}{4\pi}(g, \hat{X})_S + V(0) \qquad \text{...(4.3)}$$

subject to the condition (2.4). This solves the Neumann problem and (4.3) may be taken as analogous to Poisson's integral. We wish to seek the physical interpretation of the term $(1/4\pi)(g, \hat{X})_S$ a little later, $V(0)$ being the potential at the centre and $(1/2\pi)$ $(g, 1/r)_S$, the potential due to a single layer of density g. Here we may establish the converse proposition that (4.3) indeed solves the Neumann problem under the constraint (2.4).

Let $n_\pm$ denote the normal to S along the exterior (positive) and interior (negative) sides of S as $P \to p$, a point of S. Let, in Fig. 1 for the triangle Opq, the angle at p be α, then $\gamma = \pi - 2\alpha$. Thus

$$r = 2a \sin \tfrac{1}{2}\gamma = 2a \cos \alpha. \qquad \text{...(4.4)}$$

From (2.6) and (4.4) we have

$$r_n = r_{\dot{\rho}} = a(1 - \cos\gamma)/r = \sin\gamma/2 = \cos\alpha = r/2a \qquad \text{...(4.5)}$$

whence $$r_{n\pm} = \pm\frac{r}{2a}\,. \qquad \text{...(4.6)}$$

The following results, that are needed in the sequel, follow directly from the above:

$$\left(\frac{1}{r}\right)_{n\pm} = \mp\, 1/2ar \qquad \text{...(4.7)}$$

and

$$(\log 1/r)_{n\pm} = \mp\, 1/2a. \qquad \text{...(4.8)}$$

From the triangle OPq, by projection and cosine formulae, we have

$$\hat{X} = (1/a)\log\{4a^2/(r + a + \rho)(r + a - \rho)\}. \qquad \text{...(4.9)}$$

Differentiating partially with respect to ρ and letting $\rho \to a$, we have

$$\hat{X}_{n\pm} = \pm\,\hat{X}_{\dot{\rho}} = \pm\,\frac{1}{a}\left(\frac{1}{r} - \frac{1}{a}\right). \qquad (4.10)$$

Now using formulae dealing with the discontinuities in the normal derivative of the potential of a single layer (Kellogg 1929, p. 164), (4.3) gives

$$V_{n_+} = -\,g + \frac{1}{2\pi}(g, (1/r)_{n_+})_S + \frac{1}{4\pi}(g, \hat{X}_{n_+})_S \qquad \text{...(4.11)}$$

which by virtue of (4.7), (4.10) and (2.4) simplifies to $V_{n_+} = -g$. Similarly, the companion result $V_{n_-} = g$ can be obtained. Thus for the interior (negative) region the prescribed value g on S is obtained. For the exterior (positive) region however, negative of g may be taken. Precisely the same exists for the Dirichlet problem (Kellogg 1929, p. 243). We may summarize as:

Theorem 2 — If $g(\theta, \varphi)$ is the boundary value for the interior Neumann problem then the solution is given by (4.3) subject to the constraint (2.4) and conversely.

5. Expansion in Spherical and Ultraspherical Harmonics

One of our aims is to seek a physical interpretation of the logarithmic term in (4.3), which has been shown to be harmonic. We recall that any function with sufficient differentiability can be expressed as the sum of Newtonian potentials (Kellogg 1929, p. 219) and may argue likewise. Another fact, that logarithmic potentials of distributions over an area are limiting cases of Newtonian potentials (Kellogg 1929, p. 174), could be of some assistance. Here, we follow yet another idea, that harmonic functions can be expanded in series of spherical harmonics (Kellogg 1929, p. 251).

Let for interior points of S

$$z = \cos\gamma, \quad h = \rho/a \quad \text{and } t = \frac{a - \rho\cos\gamma}{r},$$

then, by projection theorem, $|z|$, $|h|$ and $|t|$ are < 1. We may write (4.2) as

$$\hat{X} = \frac{1}{a}\left\{\log 2a + \log\frac{1}{r} - \log(1+t)\right\} \qquad \text{...(5.1)}$$

where $\quad r = a(1 - 2hz + h^2)^{1/2}$.

Expanding the log terms further, we have

$$\log\frac{1}{r} = \log\frac{1}{a} + \sum_{n=1}^{\infty} h^n \cos n\gamma/n \qquad \text{...(5.2)}$$

and $\quad \log(1+t) = t - \tfrac{1}{2}t^2 + \tfrac{1}{3}t^3 \ldots \dfrac{(-1)^{m-1}}{m}\, t^m \ldots \qquad \text{...(5.3)}$

where $\quad t^m = (1 - hz)^m (1 - 2hz + h^2)^{-m/2}. \qquad \text{...(5.4)}$

In order to see the terms in (5.4) more clearly, we recall certain expansion results (Whittaker and Watson 1965, p. 329).

If

$$(1 - 2hz + h^2)^{-m} = \Sigma\, h^n C_n^m(z),\ C_0^m = 1, \qquad \text{...(5.5)}$$

then, for n integer,

$$C_n^m(z) = \frac{(-2)^n m(m+1)\dots(m+n-1)}{n!(2n+2m-1)(2n+2m-2)\dots(n+2m)}$$
$$\times (1-z^2)^{(1/2)-m} \times \frac{d^n}{dz^n}(1-z^2)^{n+m-(1/2)}. \qquad \dots(5.6)$$

Here $C_n^m(z)$ are the Gegenbaüer or ultraspherical polynomials (Luke 1969, p. 339). For $m = \frac{1}{2}$, we have Rodrigue's formula $C_n^{1/2} = P_n(z)$, where $P_n(z)$ are the Legendre polynomials. Thus the coefficients of h^n in (5.2) and (5.4) can be expressed in terms of $P_n(z)$.

From (5.6) we notice that $C_n^m(z)$ turn out to be linear combinations of P_i, $i \leqslant n$, i being even or odd, according as n is even or odd. For example,

$$C_1^{3/2} = 3z = 3P_1 \text{ and } C_2^{3/2} = \tfrac{3}{2}(5z^2 - 1) = (5P_2 + P_0).$$

Thus we have:

Theorem 3 — Ultraspherical harmonics $C_n^m(z)$ are linear combinations of the zonal harmonics $P_i(z)$, $i \leqslant n$, where i is even or odd, according as n is even or odd.

Since $\cos n\gamma$ is a polynomial of degree n in $\cos\gamma$, (5.2) can be expanded in terms of P_i. Thus

$$\log\frac{1}{r} = \log\frac{1}{a} + hP_1 + \frac{h^2}{6}(4P_2 - P_0) + \frac{h^3}{15}(8P_3 - 3P_1)\dots \qquad \dots(5.7)$$

Also, from (5.4) and (5.5) we have

$$t^m = (1-hz)^m\left(\sum_i h^i C_i^{m/2}(z)\right). \qquad \dots(5.8)$$

Here, ultraspherical polynomials enter as multipliers with other polynomials. Terms above can be expressed in Legendre polynomials as an interesting exercise in special functions. For example,

$$t^4 = P_0 + \tfrac{4}{3}h^2(P_2 - P_0).$$

Substituting for various values in terms of P_i in (5.1) and using the fact that

$$\log 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4}\dots$$

we may write $\hat{X}$ in a compact form as

$$\hat{X} = \frac{1}{a}\sum_{i=0}^{\infty}\left(\sum_{j=0}^{j=i}\alpha_{ij}h^i P_j\right) \qquad \dots(5.9)$$

where α_{ij} are constant coefficients. For terms up to h^3 they are given as:

$$\alpha_{00} = 0, \alpha_{10} = 0, \alpha_{11} = 1, \alpha_{20} = -\tfrac{1}{6}, \alpha_{21} = 0, \alpha_{22} = \tfrac{2}{3}, \ldots \qquad \ldots(5.10)$$

6. Spherical Harmonics : Potentials of Multipoles

We recall that every function $g(\theta, \varphi)$ which is continuous together with its derivatives up to the second order on the sphere, may be expanded in an absolutely and uniformly convergent series in terms of spherical harmonics (Courant and Hilbert 1953, p. 513) and (Whittaker and Watson 1965, p. 392). Let $g(\theta, \varphi)$, the boundary value, be such that

$$g(\theta, \varphi) = \sum_{n=1} a^n \{A_n P_n(\cos\theta) + \sum_{m=1}^{n} (A_n^{(m)} \cos m\varphi + B_n^{(m)} \sin m\varphi) \times P_n^m (\cos\theta)\} \qquad \ldots(6.1)$$

converges uniformly throughout the domain

$$0 \leqslant \theta \leqslant \pi; \; -\pi \leqslant \varphi \leqslant \pi.$$

Here, $P_n^m(\cos\theta)$ are Ferrer's associated Legendre functions.

By the addition theorem (Whittaker and Watson 1965, p. 328)

$$P_n(\cos\gamma) = P_n(\cos\theta)\, P_n(\cos\theta') + 2 \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta) \times P_n^m(\cos\theta') \cos m(\varphi - \varphi') \qquad \ldots(6.2)$$

and, from the orthogonality properties, we get

$$(g, P_n)_S = \frac{4\pi a^2}{2n+1}\, a^n S_n(\theta, \varphi) \qquad \ldots(6.3)$$

where

$$S_n(\theta, \varphi) = \{A_n P_n(\cos\theta) + \sum_{m=1}^{n} (A_n^{(m)} \cos m\varphi + B_n^{(m)} \sin m\varphi) \,.\, P_n^m(\cos\theta)\} \qquad \ldots(6.4)$$

is a surface spherical harmonic of dimension n depending upon $(2n+1)$ arbitrary linear parameters. The functions $P_n^m(\cos\theta)\cos m\varphi$ and $P_n^m(\cos\theta)\sin m\varphi$ are all linearly independent, since they are mutually orthogonal. They may be called symmetric spherical harmonics of order n.

We may write (6.3) as

$$\frac{1}{4\pi a^2}(g, h^n P_n)_S = \frac{H_n}{\lambda_n}, \; n = 1, 2, \ldots \qquad \ldots(6.5)$$

where $h = \rho/a$ and $H_n = \rho^n S_n$ are the spherical harmonics of order n, and $\lambda_n = (2n+1)$ are the corresponding eigenvalues. The above result may be treated as a generalization of Gauss' theorem of the arithmetic mean and stated as:

Theorem 4 — If $g(\theta, \varphi)$ is continuous together with its derivatives up to the second order then its weighted average over the sphere with a homogeneous polynomial of degree n is the quotient of the spherical harmonic of order n and the corresponding eigenvalue.

From the foregoing, we may write (6.1) as

$$g(\theta, \varphi) = \frac{1}{4\pi a^2} \sum \lambda_n (g, P_n)_S. \qquad \ldots(6.6)$$

For $a = 1$, it is the result employed by Power (1951) who defines $(g, P_n)_S$ as the surface spherical harmonic.

Substituting the value of $\hat{X}$ from (5.9) into the second term of (4.3), interchanging the order of summation and integration and using (6.5), we get

$$\frac{1}{4\pi}(g, \hat{X})_S = a \sum_i \frac{\beta_i}{\lambda_i} H_i, \; \alpha_{ii} = \beta_i. \qquad \ldots(6.7)$$

For the first few terms from (5.10), we have

$$\beta_1 = 1, \quad \beta_2 = 2/3, \quad \beta_3 = 8/15. \qquad \ldots(6.8)$$

Thus we have the desired expansion in terms of spherical harmonics determined by the boundary value g.

According to the Maxwell-Sylvester representation (Courant and Hilbert 1953, pp. 514-21) all spherical harmonics correspond to the potentials of 'multipoles'. The salient points of the theory are : firstly each of the $(2n+1)$ linearly independent symmetrical harmonics of order n is represented in terms of multipole potentials and is given by a sum of such potentials. Consequently, every nth order spherical harmonic is given by a sum of such potentials and finally every sum of several such multipole potentials is equal to the potential of a single multipole. Thus (6.7) represents the potential of a multipole. Furthermore, since no multipole potential can vanish identically (Courant and Hilbert 1953, p. 519), $g(\theta, \varphi)$ is non-zero.

7. Expansion in Fundamental Functions

Introducing an orthonormal set of $(2n+1)$ linearly independent eigen functions $\{\varphi_n^m, \psi_n^m\}$ corresponding to the eigenvalue $\lambda_n = (2n+1)$, we may write (6.3) as

$$(g, P_n)_S = \frac{4\pi}{\lambda_n} \sum_{m=0}^{n} (a_{nm}\varphi_n^m + b_{nm}\psi_n^m) \qquad \ldots(7.1)$$

where $a_{nm} = (g, \varphi_n^m)_S$ and $b_{nm} = (g, \psi_n^m)_S$.

Here, φ_n^m and ψ_n^m are known to correspond to the single and double layer distributions (Howland 1955) and are respectively proportional to the even and odd spherical harmonics. If we set

$$H_n^m = a_{nm}\,\varphi_n^m + b_{nm}\,\psi_n^m \qquad \text{...(7.2)}$$

we have from (6.3) and (7.1)

$$a^{n+2}S_n(\theta, \varphi) = \sum_{m=0}^{n} H_n^m \qquad \text{...(7.3)}$$

whence (6.1) gives

$$g(\theta, \varphi) = \sum a^n S_n = \frac{1}{a^2}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} H_n^m\right). \qquad \text{...(7.4)}$$

Since $\psi_n^0 = 0$, (7.3) represents a series of $(2n + 1)$ distributions, of which $(n + 1)$ are monopole and n are dipole. Also by virtue of the constraint equation (2.4) and the fact that $\varphi_0^0 = 1$ and $\psi_n^0 = 0$, we have $a_{00} = 0$ and $b_{n0} = 0$. Consequently $H_0^0 = 0$, thus from (6.5) and (7.1), the following is immediate

$$\frac{1}{4\pi a^2}(g, h^nP_n)_S = \frac{H_n}{\lambda_n} = \frac{1}{a^2}\sum_{m=0}^{n} h^nH_n^m / \lambda_n. \qquad \text{...(7.5)}$$

Employing the expansion of $1/r$ in terms of h^nP_n (Kellogg 1929, p. 251) we have

$$\frac{1}{2\pi}(g, 1/r)_S = 2a\sum_{n=0} H_n/\lambda_n. \qquad \text{...(7.6)}$$

The solution $V(P)$ in (4.3), by virtue of (6.7) and (7.6) becomes

$$V(P) = a\sum_{n=0}^{\infty}\left(\frac{2+\beta_n}{\lambda_n}\right)H_n + V(0) \qquad \text{...(7.7)}$$

which gives expansion in terms of spherical harmonics. We may write the above by using (7.5) as

$$V(P) = \frac{1}{a}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{2+\beta_n}{\lambda_n}\right)h^nH_n^m + V(0) \qquad \text{...(7.8)}$$

which gives the desired expansion in fundamental functions.

8. Convergence of Series

For points of R, $h < 1$, also the coefficients of the double series (7.8) are such that for $n > 1$, $|(2 + \beta_n)/\lambda_n| < 1$, and on S, $h = 1$, whence (7.4) gives

$$V(p) \leqslant (1/a) \sum_{n} \sum_{m=0}^{n} H_n^m + V(0) \leqslant ag(\theta, \varphi) + V(0). \qquad ...(8.1)$$

We may summarize as:

Theorem 5 -- If $g(\theta, \varphi)$ is a continuous and differentiable function on the surface of the sphere S, then the solution $V(P)$ of the Neumann problem corresponding to this as its boundary value is given by an absolutely and uniformly convergent series (7.7) when $P \in R$. The series converges absolutely and uniformly on S as well.

Corollary — $V(p)$ is bounded from above.

We may write (8.1) as

$$\frac{1}{a}(V(p) - V(0)) \leqslant g(\theta, \varphi). \qquad ...(8.2)$$

For small enough a, in the limit, equality obtains above. Thus the maximum defect of the potential at 0 and on S is linearly related with the boundary value g.

Evidently (7.7) and (8.1) give

$$V(P) \leqslant V(p) \leqslant ag(\theta, \varphi) + V(0). \qquad ...(8.3)$$

Applying Gauss' theorem of the arithmetic mean (Kellogg 1929, p. 223) over a sphere of radius ρ to the outer inequality above, we have $0 \leqslant g(\theta, \varphi)$. Similarly, treating the second inequality over S and using (2.4), equality obtains. Thus $V(P)$ is an increasing function of ρ. For a unit charge at P, $V(0)$ is a positive quantity $(1/a) + (1/\rho)$. It follows that

$$V(0) \leqslant V(P) \leqslant ag(\theta, \varphi) + V(0) \qquad ...(8.4)$$

which may be treated as an analogue of Harnack's inequality (Kellogg 1929, pp. 262-63)

$$cU(0) \leqslant U(P) \leqslant CU(0) \qquad ...(8.5)$$

obtained with the help of Poisson's integral for harmonic functions $U(P) \geqslant 0$. Here $c < 1 < C$ are certain constants. We observe that Harnack's condition that the potential be strictly positive or zero is also met in this case. If the potential is strictly negative or zero, then the inequalities would be reversed.

9. Uniqueness and Symmetric Properties

Denoting partial derivatives with subscripts and the value of functions on S with ^ as before, we see that, for $\rho = a$

$$\hat{X}_{\hat{\rho}'} = \hat{X}_{\hat{\rho}} = \frac{1}{a}\left(\frac{1}{r} - \frac{1}{a}\right) \quad \ldots(9.1)$$

where $r = 2a \cos \alpha$, α being the angle Opq (Fig. 1). We conclude that X is symmetric with respect to differentiation along the normals at p and q. Furthermore, in view of property (ii) of N, the first two terms of (3.8) must share this property, which indeed they do. Consequently N is symmetric in the sense

$$\hat{N}_{\hat{\rho}'} = (2/r)_{\hat{\rho}} + \hat{X}_{\hat{\rho}} = -1/a^2 \quad \ldots(9.2)$$

and $N_{\rho'}$ is a constant, independent of ρ. Thus:

Theorem 6 — (a) If $N(Q, P)$ is the Neumann function for the sphere given by (3.8), then

$$\hat{N}_{\hat{\rho}'} = \hat{N}_{\hat{\rho}}. \quad \ldots(9.3)$$

The above is the analogue of the symmetric property of the Green's function of the first kind.

The uniqueness of the Neumann function can now be established by arguments similar to those given for Green's function. Let $N_i(Q, P)$, $i = 1, 2$ be two such functions and $F(Q, P)$ their difference. Obviously F will have the following properties: $F_{\hat{\rho}'} = 0$, and $\nabla^2 F = 0$ in R. Applying Green's identity to N_1 and N_2, and using their properties we have

$$(F(q, P), 1)_S = 0, \; P \in R, \; q \in S. \quad \ldots(9.4)$$

Consequently, by Gauss' theorem of the arithmetic mean $F(O, P) = 0$. Recalling that an harmonic function which vanishes on the boundary of a region identically vanishes in that region (Kellogg 1929, p. 213), we conclude that $F(Q, P) = 0$ for all $P, Q \in R$. Hence $N_1 = N_2$. Thus:

Theorem 6 — (b) The Neumann function $N(Q, P)$ given by (3.8) is unique.

From this vantage position we look at the solution (4.3). Here V and X are harmonic functions. If we set $V_{\hat{\rho}'} = -g$, then for the interior case, by virtue of Green's identity, we have

$$(g, \hat{X})_S = -(V_{\hat{\rho}'}, \hat{X})_S = (V, \hat{X}_{\hat{\rho}'})_S. \quad \ldots(9.5)$$

Substituting the value of $\hat{X}_{\hat{\rho}'}$ from (9.1) and using Gauss' theorem of the arithmetic mean, we have

$$(g, \hat{X})_S = (1/a)(V, 1/r)_S - 4\pi V(0). \quad \ldots(9.6)$$

Consequently, using the above expression in (4.3) it becomes

$$V(P) = \frac{1}{2\pi}(g, 1/r)_S + \frac{1}{4\pi a}(V, 1/r)_S \quad ...(9.7)$$

which represents potential due to single layer. Such potentials are known to be continuous on the boundary (Kellogg 1929, p. 160). Thus by letting $P \to p$, we see that $V(p)$ is given by the following integral equation.

$$V(p) = \frac{1}{2\pi}(g, 1/r)_S + \frac{1}{4\pi a}(V, 1/r)_S \quad ...(9.8)$$

where r is the distance between the points p and $q \in S$. The kernel of the integral equation above is symmetric and weakly singular [Mikhlin 1960, p. 77] and Fredholm Theory (1900, 1903) is directly applicable. Thus we have:

Theorem 7 — If g is the boundary value for the interior Neumann problem, the potential on the boundary satisfies a Fredholm integral equation of the second kind (9.8) with a symmetric and weakly singular kernel.

In part II of the paper we apply function theoretic techniques (Howland 1955, 1968; Nayar 1975, 1977, 1978; Riesz and Nagy 1955) and the theory of symmetrizing kernels (Howland 1955, 1968; Nayar 1975, 1977, 1978) to the functional equation above. In the sequel we have a method to solve a system of integral equations of mixed kind, simultaneously.

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APPENDIX

A 1. *Spherical Polar System* — We now prove the harmonic nature of $N(Q, P)$ given by (3.8) by considering variation in ρ', θ', φ'. We deal with the three terms separately and see that each is harmonic.

The Laplacian in polar spherical coordinates is

$$\nabla^2 N = \rho'^2 N_{\rho'\rho'} + 2\rho' N_{\rho'} + N_{\theta'\theta'} + \cot\theta'\, N_{\theta'} + \frac{1}{\sin^2\theta'} N_{\phi'\varphi'}. \qquad ...(1)$$

For $N_1 = \frac{1}{r}$, where r is given by (1.1) and (1.3), various partial derivatives are:

$$(N_1)_{\rho'} = -(\rho' - \rho\cos\gamma)/r^3$$

$$(N_1)_{\rho'\rho'} = 2r^{-3} - 3\rho^2 \sin^2\gamma r^{-5} \qquad ...(2)$$

$$(N_1)_{\theta'} = -\rho\rho'[\cos\theta\sin\theta' - \sin\theta\cos\theta'\cos(\varphi - \varphi')]r^{-3}$$

$$(N_1)_{\theta'\theta'} = 2\rho^2\rho'^2[\cos\theta\sin\theta' - \sin\theta\cos\theta'\cos(\varphi - \varphi')]^2 r^{-5}$$
$$-\{\rho\rho'\cos\gamma r^{-3} - \rho^2\rho'^2[\cos\theta\sin\theta'$$
$$- \sin\theta\cos\theta'\cos(\varphi - \varphi')]^2 r^{-5}\} \qquad ...(3)$$

$$(N_1)_{\varphi'} = \rho\rho'\sin\theta\sin\theta'\sin(\varphi - \varphi')r^{-3}$$

and

$$(N_1)_{\varphi'\varphi'} = -\rho\rho'\sin\theta\sin\theta'\cos(\varphi - \varphi')r^{-3}$$
$$+ 3\rho^2\rho'^2\sin^2\theta\sin^2\theta'\sin^2(\varphi - \varphi')r^{-5}. \qquad ...(4)$$

Substituting various quantities in (1), we have

$$\nabla^2 N_1 = \rho\rho'[2\cos\gamma - \cos\gamma - \{\cos\theta\cos\theta' - \sin\theta\cos\theta'\cot\theta'$$
$$\times\cos(\varphi - \varphi') + \sin\theta\,\mathrm{cosec}\,\theta'\cos(\varphi - \varphi')\}]r^{-3}$$
$$+ 3\rho^2\rho'^2[-\sin^2\gamma + \{(\cos\theta\sin\theta' - \sin\theta\cos\theta'$$
$$\times\cos(\varphi - \varphi'))^2 + \sin^2\theta\sin^2(\varphi - \varphi')\}]r^{-5} \qquad ...(5)$$

the terms in the curly brackets above simplify to $\cos\gamma$ and $\sin^2\gamma$ respectively by virtue of (1.3). Consequently terms in the square brackets vanish. Thus N_1 is harmonic.

A 2. Let $N_2 = a/\rho r'$, where r' is given by (1.2) and (1.3). The corresponding partial derivatives are:

$$(N_2)_{\rho'} = -a(\rho\rho' - a^2\cos\gamma)/\rho^2 r'^3$$

$$(N_2)_{\rho'\rho'} = 2a(\rho\rho' - a^2\cos\gamma)^2(\rho^3 r'^5)^{-1} - a^5\sin^2\gamma(\rho^3 r'^5)^{-1} \qquad ...(6)$$

whence

$$\rho'^2(N_2)_{\rho'\rho'} + 2\rho'(N_2)_{\rho'} = 2a\, \rho'^2(\rho\rho' - a^2 \cos\gamma)^2 (\rho^3 r'^5)^{-1} - a^5\rho'^2 \sin^2\gamma(\rho^3 r'^5)^{-1} - 2a\rho'(\rho\rho' - a^2\cos\gamma)(\rho^2 r'^3)^{-1} \quad \ldots(7)$$

likewise,

$$(N_2)_{\theta'\theta'} + \cot\theta'(N_2)_{\theta'} = -a^3\rho'\cos\gamma\,(\rho^2 r'^3)^{-1} + 3a^5\rho'^2[\cos\theta\sin\theta' - \sin\theta\cos\theta'\cos(\varphi-\varphi')]^2(\rho^3 r'^5)^{-1} - a^3\rho'[\cos\theta\cos\theta' - \sin\theta\cos\theta'\cot\theta'\cos(\varphi-\varphi')](\rho^2 r'^3)^{-1} \quad \ldots(8)$$

and

$$\frac{1}{\sin^2\theta'}(N_2)_{\varphi'\varphi'} = 3a^5\rho'^2\sin^2\theta\sin^2(\varphi-\varphi')(\rho^3 r'^5)^{-1} - a^3\rho'\sin\theta\,\mathrm{cosec}\,\theta'\cos(\varphi-\varphi')(\rho^2 r'^3)^{-1} \quad \ldots(9)$$

adding (7), (8) and (9), and simplifying as before, we get

$$\nabla^2 N_2 = \frac{2a\rho'^2}{\rho^3 r'^5}[(\rho\rho' - a^2\cos\gamma)^2 - \rho^2 r'^2] + \frac{2a^5\rho'^2\sin^2\gamma}{\rho^3 r'^5}. \quad \ldots(10)$$

The terms in the square brackets simplify to $-a^4\sin^2\gamma$ by virtue of (1.2). The right-hand side above vanishes. Hence N_2 is harmonic

A 3. Let $D_1 = a^2 - \rho\rho'\cos\gamma + \rho r'$ ...(11)

then we may write $X(Q, P)$ from (3.9) as

$$X = (1/a)\log 2a^2 - (1/a)\log D_1 \quad \ldots(12)$$

whence

$$\rho'^2 X_{\rho'\rho'} + 2\rho' X_{\rho'} = \rho'^2[(a^2+\rho r')\cos\gamma - \rho\rho']^2(ar'^2 D_1^2)^{-1} + 2\rho'[(a^2+\rho r')\cos\gamma - \rho\rho'](ar' D_1)^{-1} - a^3\rho'^2\sin^2\gamma\,(\rho r'^3 D_1)^{-1}, \quad \ldots(13)$$

likewise,

$$X_{\theta'\theta'} + \cot\theta' X_{\theta'} = [\rho\rho' + (a^2\rho'/r')]^2[(\cos\theta\sin\theta' - \sin\theta\cos\theta' \times \cos(\varphi-\varphi'))]^2(aD_1^2)^{-1} - [\rho\rho'\cos\gamma + \rho\{a^2\rho'\cos\gamma(\rho r')^{-1} - a^4\rho'^2(\cos\theta\sin\theta' - \sin\theta\cos\theta'\cos(\varphi-\varphi'))^2(\rho^2 r'^3)^{-1}](aD_1)^{-1}$$

(equation continued on p. 1282)

$$- \cot\theta'(\rho\rho' + (a^2\rho'/r'))\{\cos\theta\sin\theta'$$
$$- \sin\theta\cos\theta'\cos(\varphi-\varphi')\}(aD_1)^{-1}$$
$$+ [\rho'^2(a^2+\rho r')^2(ar'^2D_1^2)^{-1} + a^3\rho'^2(\rho r'^3D_1)^{-1}]$$
$$\times \{\cos\theta\sin\theta' - \sin\theta\cos\theta'\cos(\varphi-\varphi')\}^2$$
$$- \rho'(a^2+\rho r')\{\cos\gamma + \cos\theta\cos\theta'$$
$$- \sin\theta\cos\theta'\cot\theta'\cos(\varphi-\varphi')\}(ar'D_1)^{-1} \quad \ldots(14)$$

and

$$(1/\sin^2\theta')X_{\varphi'\varphi'} = [\rho'^2(a^2+\rho r')^2(ar'^2D_1^2)^{-1} + a^3\rho'^2(\rho r'^3D_1)^{-1}]$$
$$\times \sin^2\theta\sin^2(\varphi-\varphi') - \rho'(a^2+\rho r')$$
$$\times \sin\theta\operatorname{cosec}\theta'\cos(\varphi-\varphi')(ar'D_1)^{-1}. \quad \ldots(15)$$

Adding (13), (14) and (15) and simplifying as before, we get

$$\nabla^2 X = \rho'^2[(a^2+\rho r')^2 - 2\rho\rho'(a^2+\rho r')\cos\gamma + \rho^2\rho'^2](ar'^2D_1^2)^{-1}$$
$$- 2\rho\rho'^2(ar'D_1)^{-1}. \quad \ldots(16)$$

The terms in the square brackets above simplify to $2\rho r'D_1$ by virtue of (11) and (1.2). Consequently the terms on the right vanish. Hence X is harmonic. Thus we have shown that each constituent term of (3.8) considered as a function of ρ', θ', φ' is harmonic, whence $N(Q, P)$ possesses property (iii) in three dimensions.