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NEUMANN FUNCTION FOR THE SPHERE—II

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In part I of this paper the Neumann function for a sphere was constructed and some of its properties, like uniqueness and symmetric nature, were established. Certain expansions in terms of fundamental functions were given and their convergence noted.

The above led to an analogue of the Poisson integral as a solution to the Neumann problem for the sphere. The surface integral was interpreted as the sum of potentials due to monopoles and 'multipoles' respectively. It was shown that, in the limiting process, the potential on the sphere satisfies a Fredholm integral equation of the second kind with a symmetric and weakly singular kernel.

Herein, we apply function theoretic techniques which yield quickly the density of the single layer as a Neumann series in terms of the boundary values, which in turn gives the desired potential.

In the sequel, we have a method to solve simultaneously a system of integral equations of the mixed kind. It is also noted that the integral equation of the second kind represents in a compact form, all the four discontinuity relations connected with potentials and their normal derivatives for double and single layers, respectively.

1. INTRODUCTION

In part I of the paper (Nayar 1981), the Neumann function for a sphere was constructed using the classical method of images which led to an analogue of the Poisson integral solving the Dirichlet problem. Following notation of our earlier work (Nayar 1977, 1978), we may write the integral as

$$V(P) = Gg + Xg + V(O) \qquad \text{...(1.1)}$$

where O is the centre of the sphere S and $P \in R$, (see Nayar 1981, Fig.1) the interior of S. $V(P)$ is the potential at P and g is the value of the normal derivative of V on S, such that

$$(g, 1)_S = 0. \qquad \text{...(1.2)}$$

The term Gg on the right of (1.1) is to be understood as integration over S with respect to $q \in S$, where

$$G(P, q) = \frac{1}{2\pi} \cdot \frac{1}{r}. \qquad ...(1.3)$$

Likewise, the second term Xg is also a surface integral, where

$$X(P, q) = \frac{1}{4\pi a} \cdot \log \frac{2a}{2a - \rho \cos \gamma + r} \qquad ...(1.4)$$

a being the radius of the sphere and r the distance between the points $P(\rho, \theta, \varphi) \in R$ and $q \in S$, such that $P\hat{O}q = \gamma$.

Using properties of X, it was shown by a limiting process that when $P \to p$, a point on S, (1.1) yields, for the potential on the sphere, a functional equation

$$V = Gg + \lambda GV, \quad \lambda = \frac{1}{2a} \qquad ...(1.5)$$

where $G = \frac{1}{2\pi} \cdot \frac{1}{r_{pq}}$, r_{pq} being the distance between $p, q \in S$.

2. Functional Equation

Equation (1.5) is Fredholm integral equation of the second kind having a symmetric and weakly singular kernel (Nayar 1977). Hence the Fredholm alternative is directly applicable (Fredholm 1900, Mikhlin 1960).

Thus a unique solution exists for the non-homogeneous equation (1.5), since we can easily establish that the corresponding homogeneous equation

$$V = \lambda GV \qquad ...(2.1)$$

has only the trivial solution.

Evidently for $g = 0$, (1.1) gives

$$V(P) = V(O) \qquad ...(2.2)$$

where P is any interior point of S and O its centre which implies that the potential is constant inside S and since it is continuous throughout, it is constant on S as well, whence (2.1) gives

$$\lambda(G, 1)_S = 1, \qquad ...(2.3)$$

which is not true. Hence (2.1) has only the trivial solution. We may summarize the above as:

Theoerm 1 — Potential on the sphere corresponding to the Neumann problem is unique.

In an earlier work (Nayar 1975), we have shown that the boundary values of the potential and its normal derivatives correspond to continuous moment and density, respectively.

Let μ be the density of the single layer which gives rise to potential $V(p)$ on S, then

$$V = G\mu. \qquad \ldots(2.4)$$

Equanting the values of V from (1.5) and (2.4), we have

$$G\mu = (I - \lambda G)^{-1} Gg \qquad \ldots(2.5)$$

since G is symmetric, we have

$$\mu = (I - \lambda G)^{-1} g \qquad \ldots(2.6)$$

whence,

$$\mu = \sum_{n=0}^{\infty} \lambda^n \mu_n, \quad \mu_0 = g \qquad \ldots(2.7)$$

where μ_n is the nth 'moment' defined as

$$\mu_n = G\mu_{n-1}. \qquad \ldots(2.8)$$

The series converges absolutely since $| \lambda G | < 1$. For small enough λ, $\mu \sim g$. Thus for spheres with large radii, the density will approximate the boundary value of the normal derivative. We may summarize as:

Theorem 2 — The functional equation (1.5) corresponding to the Neumann problem has a solution $V = G\mu$, where μ is given by an infinite Neumann series (2.7) which converges absolutely.

It is pertinent to seek physical interpretation of the 'moments' μ_n. From the foregoing it is clear that at least μ_0 and μ_1 correspond to the density of single layer and moment of double layer, respectively. The development in Part 1 of the paper suggests that μ_n, $n > 1$, must of necessity correspond to the moments of the 'multipoles'.

To see this we employ, in the next section, the theory of symmetrizing kernels (Howland 1968) and use some of the results established earlier (Nayar 1975, 1977, 1978, 1981).

3. Integral Equations of the Mixed Kind

We recall that, if the solutions of the Dirichlet and Neumann problems are sought in terms of continuous density and moment respectively, we are led to the integral equations of the first kind (Howland 1968)

$$f = G\mu \qquad \ldots(3.1)$$

$$g = D\gamma \qquad \ldots(3.2)$$

where f and g are the values of the potential and its normal derivative on S; μ and γ denote the density and moment of the distributions, respectively.

$$G(p, q) = \frac{1}{2\pi} \cdot \frac{1}{r_{pq}} \quad \text{and} \quad D(p, q) = \frac{\partial^2}{\partial n_p \partial n_q} G(p, q)$$

are symmetric transformations that are the left symmetrizers of the kernel $K = \frac{\partial}{\partial n_p} G(p, q)$ and $K^* = \frac{\partial}{\partial n_q} G(p, q)$, p and $q \in S$, and n_p is the outward normal at p. K and K^* are associated with the Neumann and Dirichlet problems, respectively (Nayar 1978). If K^2 and K^{*2} are the second iterates, then these transformations are connected by the relations

$$DG = K^2 - I, \quad GD = K^{*2} - I \qquad \ldots(3.3)$$

where I is the identity transformation. Consequently, (3.1) and (3.2) are equivalent to the integral equations of the second kind

$$(K^2 - I)\mu = Df \qquad \ldots(3.4)$$

$$(K^{*2} - I)\gamma = Gg. \qquad \ldots(3.5)$$

We have shown (Nayar 1975) that the boundary values of the potential and its normal derivative correspond to continuous moment and density, respectively.

We defined in (Nayar 1978) a space H of continuous functions and endowed it with Dirichlet norm G and another norm D with respect to which K and K^* were symmetric and continuous. G, D, K and K^* are transformations of $H \to H$.

If $\mathscr{D}_G$ and $\mathscr{D}_D$ denote the domains of G and D, respectively and then if $f \in \mathscr{D}_D$, (3.4) possesses a unique continuous solution (Howland 1968). We may likewise argue from (3.5) that if $g \in \mathscr{D}_G$, then it has a unique continuous solution.

Since G-norm and D-norm are linearly comparable (Nayar 1978), we conclude that

$$\mathscr{D}_D \subseteq \mathscr{D}_G. \qquad \ldots(3.6)$$

From this vantage point we turn to the integral equation (1.5) and identify the potential $V(p)$ with the moment γ and solve integral equations (1.5), (3.1) and (3.2) of the mixed type, simultaneously. Equating the value of γ from (1.5) and (3.2), we have

$$\gamma = (I - \lambda G)^{-1} Gg = D^{-1} g. \qquad (3.7)$$

Consequently,

$$D^{-1} = (I - \lambda G)^{-1} G. \qquad \ldots(3.8)$$

Whence

$$G^{-1} = D + \lambda. \qquad \ldots(3.9)$$

This establishes the existence of G^{-1} as a linear transformation in terms of D and the parameter λ. Also D^{-1} exists as an infinite series of linear transformations.

Let $\mu_0 = g$ and $\gamma_0 = G\mu_0 = f$, then (3.7) gives

$$\gamma = (I - \lambda G)^{-1} \gamma_0. \qquad \ldots(3.10)$$

Using (3.8), we have

$$\gamma = D^{-1} G^{-1}\gamma_0. \qquad \ldots(3.11)$$

Substituting the value of G^{-1} from (3.9) above, we have,

$$\gamma = (I + \lambda D^{-1}) \gamma_0. \qquad \ldots(3.12)$$

For small enough λ, $\quad \gamma \sim \gamma_0 = Gg.$

Since all the solutions (Howland 1968) are of the form $\gamma = G\mu$ we get

$$\mu = (I + \lambda D^{-1}) \mu_0. \qquad \ldots(3.13)$$

Comparing it with the Neumann series solution (2.7), we have

$$D^{-1}g = \mu_1 + \lambda\mu_2 + \lambda^2\mu_3 + \ldots. \qquad \ldots(3.14)$$

By the very definition (2.8), $\mu_i \in \mathscr{D}_G$. We can see now that they lie in the domain of D as well. Since $\mu_1 = \gamma_0$, $\mu_2 = G\gamma_0$, whence $G^{-1}\mu_2 = \gamma_0$ and using (3.9), we get

$$(D + \lambda) \mu_2 = \gamma_0. \qquad \ldots(3.15)$$

In order that this be valid $\mu_2 \in \mathscr{D}_D$. Consequently the moments μ_i lie in the domain of D. Hence μ_i could be looked upon as moments of double distribution. Let

$$T = D + \lambda. \qquad \ldots(3.16)$$

Since D is symmetric, so is T. Evidently,

$$\mu_0 = T^n\mu_n. \qquad \ldots(3.17)$$

Recall that it possesses for $n \geqslant 1$, one and only one solution μ_n belonging to the subspace of continuous functions (Riesz and Nagy 1955, p. 183) when ever μ_0 belongs to it, whence $\mu_i \in \mathscr{D}_G$.

From (3.9) and (3.13) by virtue of the relation $\gamma_0 = Gg$, we have

$$\mu = \mu_0 + \lambda\gamma_0 + \lambda^2 D^{-1}\gamma_0. \qquad \ldots(3.18)$$

We can easily see that the 'moments' $\mu_i \in \mathscr{D}_D$ as well. Since

$$\mu_1 = \gamma_0 \text{ and } \mu_2 = G\mu_1 = (D + \lambda)^{-1}\gamma_0,$$

by virtue of (3.9), which is equivalent to

$$\lambda \mu_2 = (I - DG)\gamma_0. \quad \ldots(3.19)$$

Consequently,

$$\lambda^n \mu_{n+1} = (I - DG)^n \gamma_0. \quad \ldots(3.20)$$

Let

$$L = I - DG \quad \ldots(3.21)$$

then we may write the above as

$$\lambda^n \mu_{n+1} = L^n \gamma_0. \quad \ldots(3.22)$$

Evidently L is linear, in fact

$$L = \lambda G. \quad \ldots(3.23)$$

In the next section, for the particular case of a sphere, this will be identified with K or K^*. We may summarize the above results as:

Theorem 3 — The Neumann series solution (2.7) is such that the 'moments' $\mu_i \in \mathcal{D}_D \subseteq \mathcal{D}_G$. μ_i could be identified with the moments of the multipoles.

4. Interior and Exterior Problems

Let $\mu_\pm$ and $\gamma_\pm$ indicate the corresponding values for the exterior (positive) and interior (negative) sides of S. Considering the outward normal to the surface as positive, in the particular case of the sphere of radius a, for the interior region, we have,

$$K = K^* = -\lambda G. \quad \ldots(4.1)$$

The integral equation (1.5) then represents, in a compact form, all the four discontinuity relations in the potentials and their normal derivatives, corresponding to the double and single layers, respectively (Kellogg 1929, p. 309). Let as before (Nayar 1975),

$$V = \gamma = G\mu \quad \ldots(4.2)$$

and

$$V_{n_\pm} = \mu_\pm = \mp g. \quad \ldots(4.3)$$

Then by (4.1) and (1.5), we have, for the interior region,

$$(K + I)\mu = g = \mu_- = V_{n_-}. \quad \ldots(4.4)$$

For the exterior region, since

$$K = K^* = \lambda G \quad \ldots(4.5)$$

it becomes

$$(K - I)\mu = -g = \mu_+ = V_{n+}. \qquad \ldots(4.6)$$

We note that (4.4) and (4.6) are precisely the discontinuity relations in the normal derivative of a potential due to a single layer of density μ.

Identifying $V = \gamma$, we may write (1.5) in the form

$$(I - \lambda G)\gamma = Gg. \qquad \ldots(4.7)$$

For the interior case by (4.1) it becomes

$$(K^* + I)\gamma = Gg. \qquad \ldots(4.8)$$

If W represents the potential of a double layer of moment γ and if we identify W_+ with Gg, then (4.8) is precisely the discontinuity relation for the exterior Dirichlet problem (Kellogg 1929, p. 309). Thus (4.4) and (4.8) build up an instant connection between the interior Neumann and exterior Dirichlet problems.

We have incidently the analogue of the two theorems wherein the interior Dirichlet problem is connected with the exterior Neumann problem (Kellogg 1929, p. 311). We may summarize as:

Theorem 4 — (I) The Neumann problem for the sphere is solvable for the finite region R for any continuous values of the normal derivative g on the boundary with the constraint $(g, 1)_S = 0$.

(II) The Dirichlet problem for the sphere is solvable for the infinite region R' for any continuous boundary values.

The solutions are given as series of potentials due to monopoles and multipoles.

Finally, for the exterior case, the integral equation (1.5) becomes

$$(K^* - I)\gamma = -Gg = G\mu_+ \qquad \ldots(4.9)$$

which is precisely the discontinuity relation for the interior Dirichlet problem (Kellogg 1929, p. 309), if we identify $G\mu_+$ with W_-. Thus we have

$$W_\pm = G\mu_\mp = \pm\, Gg. \qquad \ldots(4.10)$$

Identifying $W_\pm$, the limiting values of W with the moment of double layer $\gamma_\mp$, we have

$$\gamma_\pm = G\mu_\pm = \mp Gg. \qquad \ldots(4.11)$$

The integral equations for the interior and exterior Neumann problem may be written as

$$(I \mp \lambda G)\mu = \mu_\mp = \pm\, g. \qquad \ldots(4.12)$$

For the homogeneous problem, in the interior case, using (4.4), we have

$$K\mu + \mu = 0. \quad \ldots(4.13)$$

It follows that -1 is a characteristic of the kernel K. It is also the characteristic of K^*, since

$$GK\mu + G\mu = K^*G\mu + G\mu = K^*\gamma + \gamma = 0. \quad \ldots(4.14)$$

Here we have used the symmetrizing property of G (Howland 1968), the result holds in the general case also (Kellogg 1929, p. 312).

Subtracting the two results in (4.12), we have for the exterior case

$$K\bar{\mu} + \bar{\mu} = 0 \quad \ldots(4.15)$$

where

$$\bar{\mu} = g = \tfrac{1}{2}(\mu_- - \mu_+).$$

By virtue of the discontinuity relations $\bar{\mu} = \mu$. Hence (4.13) and (4.15) are identical. Consequently, if $g = 0$, both μ and γ vanish. Thus for the exterior case eqns. (4.6) and (4.9) are such that their corresponding homogeneous equations have null solutions. This, incidently, confirms that $\lambda = 1$ is not a characteristic of K and K^* (Kellogg 1929, p. 311).

Since, from (3.21) and (3.23),

$$\lambda G = (I - DG) = L$$

we see that

$$Lg = \tfrac{1}{2}(\mu_+ - \mu_-). \quad \ldots(4.16)$$

Similarly, treating (4.12) as corresponding to the exterior and interior Dirichlet problem, we have

$$(I \pm K^*)\gamma = \gamma_\mp = G\mu_\mp = \pm Gg. \quad \ldots(4.17)$$

Subtracting the equations above, we have

$$Gg = \tfrac{1}{2}(\gamma_- - \gamma_+) \quad \ldots(4.18)$$

which conforms with the discontinuity relations (Kellogg 1929, p. 168), since by virtue of (4.10) it is equivalent to

$$\gamma_0 = \tfrac{1}{2}(W_+ - W_-). \quad \ldots(4.19)$$

Finally, we may reconcile the above results with equations (3.3). By virtue of (4.12), we have

$$D\gamma = DG\mu = (K^2 - I)\,\mu = (K - I)\,\mu_-. \quad \ldots(4.20)$$

For the exterior case it becomes

$$D\gamma_+ = (\lambda G - I)\,\mu_- = \lambda\gamma_- - \mu_-.$$

Also, using (3.9), we have

$$D\gamma_+ = (G^{-1} - \lambda)\gamma_+ = \mu_+ - \lambda\gamma_+.$$

Consequently,

$$\mu_+ - \lambda\gamma_+ = -\,(\mu_- - \lambda\gamma_-) \qquad \text{...(4.21)}$$

which is true because of (4.3) and (4.11). Conversely, we may arrive at (3.3) with the help of integral equations (1.5), (3.1) and (3.2). For the exterior case,

$$DG\mu = D\gamma = -g = (\lambda G - I)\,\mu = (\lambda G - I)\,G^{-1}\gamma_-$$

$$= (\lambda G - I)\,(D + \lambda)\,\gamma_- = (\lambda G - I)\,(\lambda G + I)\,\mu_- = (K^2 - I)\,\mu.$$

Here we have used relations (3.9) and (3.2) for G^{-1} and $D\gamma$ respectively, along with (4.5).

Also since $K = K^*$ and G and D are symmetric, we can easily get the second companion result of (3.3).

To conclude, we observe that the simultaneous solution of the system of integral equations of the mixed kind, two of which are of the first kind and one is of the second kind, has been obtained as Neumann series for μ and γ, respectively. This has been done under the premise that the boundary value corresponding to the Neumann problem $g \in \mathscr{D}_G$ and that all the solutions of the system are of the type

$$\gamma = G\mu = V \in \mathscr{D}_D.$$

Relations (3.3) between linear transformations G, D, K and K^* hold the key to the above development.

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