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ON A PARTITION-PROBLEM OF ERDÖS

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In his letter of June 3, 1981 from Mysore, Professor Erdös posed the following interesting problem:

Let $a_1^{(j)} + a_2^{(j)} + a_3^{(j)} = n, \quad 1 \leqslant j \leqslant k;$

be k partitions of n into three distinct parts. Assume further that the $3k$ integers a are all distinct. How large can k be for any given positive integer n? In this short paper, the problem is generalised and completely solved.

1. INTRODUCTION

In his letter of June 3, 1981 from Mysore, Professor P. Erdös posed the following interesting problem:

$$\text{Let} \quad a_1^{(j)} + a_2^{(j)} + a_3^{(j)} = n, \quad 1 \leqslant j \leqslant k;$$

be k partitions of n into three distinct summands. Assume further that the $3k$ integers a are all distinct. How large can k be for any given positive integers n ?

A direct generalisation of this problem of Erdös will be:

$$\text{Let} \quad a_1^{(j)} + a_2^{(j)} + a_3^{(j)} + \ldots + a_r^{(j)} = n, \quad 1 \leqslant j \leqslant k;$$

be k partitions of n into r ($\geqslant 2$) distinct parts. Assume further that the rk summands a are all distinct. Then the problem is to find the largest value of k for any given n.

The problem can also be stated in the following form:

In a pack of n cards, the first card is marked 1, the second 2, the third 3, ..., and the nth n. At the most how often can r cards be taken out of the pack and put aside if the sum of the markings on the r cards has to be n each time ?

In this short paper, we give a simple solution of this problem.

In what follows, we use the word partitions for partitions satisfying the conditions of the problem; $[x]$ stands for the largest integer in x; and u is written for $[k/2]$.

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2. The Basic Move Towards Solution

Since the kr integers a in the problem are all distinct, their sum kn cannot be less than the sum of the first kr natural numbers. Hence we must have

$$kn \geqslant kr(kr+1)/2, \quad \text{i.e. } n \geqslant (kr^2+r)/2. \qquad \text{...(2.1)}$$

The least integer n_k satisfying this condition is

$$n_k = n_k(r) = [(kr^2+r+1)/2]. \qquad \text{...(2.2)}$$

We assert that for this n_k, k partitions of the desired type do exist (and there could be no more). Three cases arise.

2.1. *The Case when r is Even.*

The partitions are given by

$$a_t^{(j)} = (t-1)k + j \text{ or } tk - j + 1; \qquad \text{...(2.3)}$$

according as t is odd or even; and $1 \leqslant t \leqslant r$, $1 \leqslant j \leqslant k$.

Proof : Evidently the a's are all distinct. Moreover

$$(a_1^{(j)} + a_2^{(j)}) + (a_3^{(j)} + a_4^{(j)}) + \dots + (a_{r-1}^{(j)} + a_r^{(j)})$$

$$= (2k+1) + (6k+1) + (10k+1) + \dots + ((2r-2)k+1)$$

$$= \tfrac{1}{2}(kr^2+r) = n_k.$$

Since this result is independent of j, our assertion holds for all even values of r.

Note that the largest summand that occurs in our partitions is

$$a_r^{(1)} = rk. \qquad \text{...(2.4)}$$

Example : For $k=5$, $r=6$, the partitions of $n_5(6)$ are

$1+10+11+20+21+30=93;$

$2+9+12+19+22+29=93;$

$3+8+13+18+23+28=93;$

$4+7+14+17+24+27=93;$

$5+6+15+16+25+26=93.$

2.2. *The Case r = 3*

The partitions of $n_k(3) = [(9k+4)/2]$ are given by

$$a_1^{(j)} = j, \quad 1 \leqslant j \leqslant k;$$

$$a_2^{(j)} = \begin{cases} 2(k-j)+1 & \text{if } 1 \leqslant j \leqslant u; \\ 2(k-j+u)+2 & \text{if } u+1 \leqslant j \leqslant k; \end{cases}$$

$$a_3^{(j)} = \begin{cases} 2k+u+j+1 & \text{if } 1 \leqslant j \leqslant u, \\ 2k-u+j & \text{if } u+1 \leqslant j \leqslant k. \end{cases}$$

PROOF : The $3k$ integers a are readily seen to be all distinct, and we also have

$$a_1^{(j)} + a_2^{(j)} + a_3^{(j)} = 4k + u + 2 = [(9k+4)/2];$$

for each $j \leqslant k$. Our assertion holds, therefore, in this case as well. The largest summand that appears in the partitions is given by

$$a_3^{(u)} = 2k + 2u + 1 = 3k \text{ or } 3k+1, \qquad \ldots(2.5)$$

according as k is odd or even.

Example : For $k = 7$, because $u = 3$, we have the partitions:

$1 + 13 + 19 = 33;$

$2 + 11 + 20 = 33;$

$3 + 9 + 21 = 33;$

$4 + 14 + 15 = 33;$

$5 + 12 + 16 = 33;$

$6 + 10 + 17 = 33;$

$7 + 8 + 18 = 33.$

Similarly, the partitions for $k = 8$, are

$1 + 15 + 22 = 38;$

$2 + 13 + 23 = 38;$

$3 + 11 + 24 = 38;$

$4 + 9 + 25 = 38;$

$5 + 16 + 17 = 38;$

$6 + 14 + 18 = 38;$

$7 + 12 + 19 = 38;$

$8 + 10 + 20 = 38.$

It is note-worthy that for odd k, all the integers from 1 to $3k$ appear as summands in our partitions. For even k, however, $2k + u + 1$ fails to appear as a summand while all other integers from 1 to $3k + 1$ do appear.

2.3. *The Case r odd and* > 3

In this case, we write

$$r = (r - 3) + 3.$$

Since $(r - 3)$ is even, we first write down the partitions for $n_k(r - 3)$. The largest summand that appears in these partitions is known to be $(r - 3)\,k$. To each summand in the partitions for $n_k(3)$, we now add the number $(r - 3)\,k$. This makes each of these summands distinct from those in the partitions for $n_k(r - 3)$. We now juxtapose these latter partitions to the right of the partitions for $n_k(r - 3)$ to get partitions of

$$n_k(r - 3) + 3(r - 3)\,k + n_k(3)$$

which is $= \frac{1}{2}\{k(r - 3)^2 + (r - 3)\} + 3(r - 3)\,k + [(9k + 4)/2]$

$$= [(kr^2 + r + 1)/2] = n_k(r).$$

These are the required k partitions.

An example will make the procedure clear.

Take $r = 7$, $k = 6$.

we have $7 = 4 + 3$.

The partitions for $n_6(4)$ are:

$$1 + 12 + 13 + 24 = 50;$$
$$2 + 11 + 14 + 23 = 50;$$
$$3 + 10 + 15 + 22 = 50;$$
$$4 + 9 + 16 + 21 = 50;$$
$$5 + 8 + 17 + 20 = 50;$$
$$6 + 7 + 18 + 19 = 50.$$

The partitions for $n_6(3)$ are:

$$1 + 11 + 17 = 29;$$
$$2 + 9 + 18 = 29;$$
$$3 + 7 + 19 = 29;$$
$$4 + 12 + 13 = 29;$$
$$5 + 10 + 14 = 29;$$
$$6 + 8 + 15 = 29.$$

Adding 24 to each of the summands in the latter set of partitions, we get

$$25 + 35 + 41 = 101;$$
$$26 + 33 + 42 = 101;$$
$$27 + 31 + 43 = 101;$$
$$28 + 36 + 37 = 101;$$
$$29 + 34 + 38 = 101;$$
$$30 + 32 + 39 = 101.$$

Juxtaposition now gives the partitions:

$$1 + 12 + 13 + 24 + 25 + 35 + 41 = 151;$$
$$2 + 11 + 14 + 23 + 26 + 33 + 42 = 151;$$
$$3 + 10 + 15 + 22 + 27 + 31 + 43 = 151;$$
$$4 + 9 + 16 + 21 + 28 + 36 + 37 = 151;$$
$$5 + 8 + 17 + 20 + 29 + 34 + 38 = 151;$$
$$6 + 7 + 18 + 19 + 30 + 32 + 39 = 151;$$

of 151 and 151 is just our $n_6(7)$.

If r is odd and $\geqslant 7$, juxtaposition can still be managed if we write $r = 3 + (r - 3)$. Thus in the above example we get the partitions:

$$1 + 11 + 17 + 20 + 29 + 30 + 43 = 151;$$
$$2 + 9 + 18 + 21 + 28 + 31 + 42 = 151;$$
$$3 + 7 + 19 + 22 + 27 + 32 + 41 = 151;$$
$$4 + 12 + 13 + 23 + 26 + 34 + 39 = 151;$$
$$5 + 10 + 14 + 24 + 25 + 35 + 38 = 151;$$
$$6 + 8 + 15 + 16 + 33 + 36 + 37 = 151.$$

See how the missing summand from the partitions of 29 has come to our help in getting the desired partitions. The largest summand utilized is still 43 as it was in the earlier case.

We might note here that the main characteristic of our set of k partitions for n_k is that the largest summand appearing in the set is the least possible. In no set of k partitions of n_k can the largest summand be less than the summand in our set.

3. The Formula for $k_n(r)$

Having already shown that for $n_k = n_k(r)$, there always exist k and only k partitions of the desired type, we now show that this is true also for each n for which

$$n_k < n < n_{k+1}.$$

Proof : Since $n < n_{k+1}$, it cannot have more than k partitions.

Let $n = n_k + m$ where $1 \leqslant m < n_{k+1} - n_k$.

Then since from each of the k partitions of n_k, we can obtain a partition of n by increasing the largest summand in the partition of n_k by m, our assertion follows.

We have thus shown that for each n from n_k to $n_{k+1} - 1$, there exist exactly k partitions.

We can now obtain our formula for $k_n = k_n(r)$ as follows:
We have

$$(kr^2 + r)/2 \leqslant n < [((k + 1) r^2 + r + 1)/2]$$
$$\leqslant ((k + 1) r^2 + r - 1)/2.$$

Hence

$$k \leqslant (2n - r)/r^2 \leqslant (k + 1) - \frac{1}{r^2} < (k + 1).$$

We must, therefore, have

$$k_n(r) = [(2n - r)/r^2].$$

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