Indian J. pure appl. Math., 12(11) : 1312–1318, November 1981

ON GENUSES OF COMPACT RIEMANN SURFACES ADMITTING SOLVABLE AUTOMORPHISM GROUPS

B. P. CHETIYA

Department of Mathematics, Gauhati University, Gauhati 781014

(*Received* 30 *March* 1981; *after revision* 30 *May* 1981)

Let m be a non-multiple of 3 greater than or equal to 4 and $g = \frac{1}{2}(m-1)(m-2)$. We prove that for every positive integer n, there is a compact Riemann surface of genus $\frac{1}{2}m(m-3)n^{2g} + 1$ which admits a solvable automorphism group G of order $6m^2n^{2g}$ such that the fourth derived group of G is identity.

1. INTRODUCTION

It has been known for a long time (see Hurwitz 1892) that the set of all biholomorphic self-transformations, usually called 'automorphisms' of a compact Riemann surface S of genus not less than 2 forms a finite group $A(S)$ called the 'automorphism group of S'. On the other hand it is also known (see Burnside 1955, Greenberg 1960) that every finite group is representable as an automorphism group of a compact Riemann surface of some genus not less than 2. Moreover, a finite group G is representable as an automorphism group of a compact Riemann surface of genus g if and only if there exists a Fuchsian group Γ and an epimorphism $\varphi : \Gamma \to G$ such that ker φ is a surface group, i.e. a Fuchsian group without elements of finite order except identity, of genus g (see Macbeath 1961b). Following Maclachlan's terminology (see Maclachlan 1965) we call G a 'surface-kernel factor group of Γ' and the corresponding homomorphism φ a 'surface-kernel homomorphism'. It is to be noted that the genus of the kernel is the genus of a Riemann surface of which G is the automorphism group.

Hurwitz (1892) proved that for any compact Riemann surface S of genus $g \geqslant 2$, the order of its automorphism group $A(S)$ cannot exceed $84(g-1)$. A finite group of order $84(g-1)$ representable as an automorphism group of a compact Riemann surface of genus $g \geqslant 2$ will be called a 'Hurwitz group'. Macbeath (1961a) proved that a finite group is a Hurwitz group if and only if it is a surface-kernel factor group of the Fuchsian triangle group (2, 3, 7). He further showed that for infinitely many values of g, a compact Riemann surface of genus g admits or does not admit a Hurwitz group, i.e. a maximal automorphism group. Being a homomorphic image of (2, 3, 7) a Hurwitz group is perfect, i.e. a Hurwitz group coincides with its derived group. This means that Hurwitz groups cannot be solvable.

The second maximal order that an automorphism group of a compact Riemann surface of genus $g \geqslant 2$ can attain is $48(g-1)$. A finite group of order $48(g-1)$ representable as an automorphism group of a compact Riemann surface of genus g will be called an 'M_2-group'. As in the case of Hurwitz groups, it can be proved (see Macbeath 1961b) that for infinitely many values of g, a compact Riemann surface of genus g admits or does not admit an M_2-group. A finite group is an M_2-group if and only if it is a surface-kernel factor group of the Fuchsian triangle group (2, 3, 8). A homomorphic image of (2, 3, 8) may not be perfect, and this indicates the possibility of the existence of solvable M_2-groups. It is known (see Chetiya 1971) that for each positive integer n, there is a compact Riemann surface of genus $2n^6+1$ which admits a solvable automorphism group of order $96n^6$ such that $G > G' > G'' > G''' > G^{iv} = \{1\}$, G''' being equal to G^{iv} only when $n = 1$. Our aim here is to generalise this result by determining the solvable surface-kernel factor groups of all Fuchsian triangle groups $(2, 3, 2m)$ where m is a non-multiple of 3 greater than or equal to 4. The necessity of the restrictions on m will be explained in §2 and in Lemma 3.1. Towards the end of this paper we prove Theorem 3.2 which will give Chetiya's result (1971) as an immediate corollary on putting $m = 4$.

2. Preliminaries

An infinite group Γ generated by k elements $x_1, x_2, \ldots, x_k$ of finite order and 2γ elements $a_1, b_1, \ldots, a_\gamma, b_\gamma$ of infinite order satisfying the relations

$$x_1^{m_1} = \ldots = x_k^{m_k} = \prod_{i=1}^{k} x_i \prod_{i=1}^{\gamma} [a_i, b_i] = 1$$

where $[a_i, b_i]$ denotes the commutator of a_i, b_i, is called a 'Fuchsian group' if

$$2\gamma - 2 + \sum_{i=1}^{k}\left(1 - \frac{1}{m_i}\right) > 0 \qquad \ldots(2.1)$$

and then Γ is said to have a presentation

$$\langle\, x_1, \ldots, x_k\, a_1, b_1, \ldots, a_\gamma, b_\gamma : x_1^{m_1} = \ldots = x_k^{m_k} = \prod_{i=1}^{k} x_i \prod_{i=1}^{\gamma} [a_i, b_i] = 1 \rangle.$$

Such a Fuchsian group Γ is usually denoted by $(\gamma : m_1, \ldots, m_k)$. The non-negative integer γ is called the 'genus' of Γ. If $\gamma = 0$, we simply use the symbol $(m_1, \ldots, m_k)$. A Fuchsian group which does not contain elements of finite order except identity is called a 'surface group'. Thus a surface group is generated by $2g$ elements of infinite order $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ satisfying the relation

$$\prod_{i=1}^{g} [\alpha_i, \beta_i] = 1,$$

where g is the genus of the group.

If $\Gamma = (\gamma; m_1, ..., m_k)$, then we use $\Delta(\Gamma)$ to denote the expression

$$2\pi \{2\gamma - 2 + \sum_{i=1}^{k} (1 - (1/m_i))\}.$$

In particular, if K is a surface group of genus g, then $\Delta(K) = 4\pi(g - 1)$. Moreover if Γ_1 is a subgroup of Γ of finite index, than it is known (see Macbeath 1961b) that

$$[\Gamma : \Gamma_1] = \frac{\Delta(\Gamma_1)}{\Delta(\Gamma)}. \qquad ...(2.2)$$

As mentioned in §1, if $\Gamma = (\gamma; m_1, ..., m_k)$ and K is a normal surface subgroup of Γ of genus g, then the surface-kernel factor group Γ/K is an automorphism group of a compact Riemann surface of genus g, and

$$|\Gamma/K| = [\Gamma : K] = \frac{\Delta(K)}{\Delta(\Gamma)} = \frac{4\pi(g-1)}{2\pi \{2\gamma - 2 + \sum_{i=1}^{k} (1 - (1/m_i))\}} \qquad ....(2.3)$$

A Fuchsian group of the form (m_1, m_2, m_3) is called a 'Fuchsian triangle group'. Let us consider the Fuchsian triangle groups $(2, 3, n)$. In view of (2.1) we immediately see that $n \geqslant 7$. It can be proved easily that $(2, 3, n)$, $n \geqslant 7$, cannot admit solvable surface-kernel factor groups if n is not a multiple of at least one of 2 and 3. So, 8 is the least value of n for which $(2, 3, n)$ may admit solvable surface-kernel factor groups. Here we consider n to be a multiple of 2 but a non-multiple of 3 greater than or equal to 4. So let us take $n = 2m$ where $3 \nmid m$ and $m \geqslant 4$, and $\Gamma = (2, 3, 2m)$. Then $(2, 3, 8)$ whose surface-kernel factor groups occur as second maximal automorphism groups of Riemann surfaces, is the first of the sequence of the Fuchsian triangle groups $(2, 3, 2m)$. Our aim in this paper is to prove the existence of an infinite number of solvable surface-kernel factor groups of $\Gamma = (2, 3, 2m)$, and hence to show that for every positive integer n their exists a solvable group G of order $6m^2n^{2g}$ where $g = \frac{1}{2}(m - 1)(m - 2)$ acting on a compact Riemann surface of genus $\frac{1}{2}m(m - 3)\,n^{2g} + 1$ such that the fourth derived group of G is the identity.

3. Solvable Surface-Kernel Factor Groups of (2, 3, 2m)

We now proceed to find an infinite family of solvable surface-kernel factor groups of $(2, 3, 2m)$ where $m \geqslant 4$ and $3 \nmid m$. For this we need the following lemmas.

Lemma 3.1 — If $\Gamma = (2, 3, 2m), m \geqslant 4$, $3 \nmid m$, then $\Gamma' = (3, 3, m)$, $\Gamma'' = (m, m, m)$ and Γ''' is a surface group of genus $g = \frac{1}{2}(m - 1)(m - 2)$. Moreover $\Gamma/\Gamma' \cong Z_2$, $\Gamma'/\Gamma'' \cong Z_3$, $\Gamma''/\Gamma''' \cong Z_m \oplus Z_m$ and $\Gamma'''/\Gamma^{\text{iv}} \cong Z \oplus Z \oplus \dots \oplus Z$ with $2g$ summands.

Proof: Let $\Gamma = \langle x, y : x^2 = y^3 = (xy)^{2m} = 1 \rangle$. Then Γ/Γ' is generated by u, v satisfying $u^2 = v^3 = (uv)^{2m} = 1$, $uv = vu$ which give $v = 1$. Thus $\Gamma/\Gamma' \cong Z_2$. Now

$$\Delta(\Gamma) = 2\pi\left\{-2 + \tfrac{1}{2} + \tfrac{2}{3} + \frac{2m-1}{2m}\right\} = 2\pi\left(\tfrac{1}{6} - \frac{1}{2m}\right)$$

and then

$$\Delta(\Gamma') = 2\Delta(\Gamma) = 4\pi\left(\tfrac{1}{6} - \frac{1}{2m}\right) < \frac{2\pi}{3}$$

so that $\Delta(\Gamma') < \pi$. This shows that the genus of Γ' is zero, because Γ' may have positive genus only if $\Delta(\Gamma') \geqslant \pi$.

Now let $\varphi : \Gamma \to Z_2$ be the abelianising homomorphism such that $\ker \varphi = \Gamma'$. Then $x \notin \Gamma'$, for otherwise

$$\varphi(\Gamma) = \langle\, \varphi(x), \varphi(y)\,\rangle \neq Z_2.$$

But $y \in \Gamma'$, for otherwise

$$\varphi(\Gamma) = \langle\, \varphi(x), \varphi(y)\,\rangle \neq Z_2.$$

It is known (see Macbeath 1961b) that the finite order elements of Γ, and therefore of Γ', are conjugates of powers of x, y, xy. We see that

$$x \notin \Gamma' \Rightarrow xy \in \Gamma'.$$

Since $y \in \Gamma'$, $xyx \in \Gamma'$. Thus $xyx.\ \ y = (xy)^2 \in \Gamma'$. Since $x \notin \Gamma'$, no conjugate of x is in Γ', for

$$\alpha x \alpha^{-1} \in \Gamma' \Rightarrow \alpha x \in \Gamma'\alpha = \alpha\Gamma'$$

$$\Rightarrow x \in \Gamma', \text{ a contradiction.}$$

Since x has order 2, conjugates of no power ($\neq 1$) of x are in Γ'. A power of y is either identity or of order 3, and so is any conjugate of a power of y. Finally, a conjugate of any power of $(xy)^2$ can be expressed in terms of conjugates of $(xy)^2$. Therefore Γ', its genus being zero, is generated only by finite order elements which are conjugates of y and $(xy)^2$, i.e. Γ' is generated by elements of order 3 and m. Let Γ' be generated by s elements of order 3 and t elements of order m. We then get

$$4\pi\left(\tfrac{1}{6} - \frac{1}{2m}\right) = 2\pi\left\{-2 + \frac{2s}{3} + \frac{m-1}{m}.\, t\right\}$$

i.e.

$$\tfrac{1}{3} - \frac{1}{m} = -2 + \frac{2s}{3} + \left(1 - \frac{1}{m}\right)t.$$

For a fixed m, the above diophantine equation has the only solution $s = 2$, $t = 1$. Thus $\Gamma' = (3, 3, m)$.

Now let Γ' be generated by x_1, y_1 satisfying $x_1^3 = y_1^3 = (x_1y_1)^m = 1$. Then Γ'/Γ'' is generated by u_1, v_1 satisfying

$$u_1^3 = v_1^3 = (u_1v_1)^m = 1,\ u_1v_1 = v_1u_1.$$

Since 3 does not divide m, $u_1v_1 = 1$, i.e. $u_1 = v_1^{-1}$. Then $\Gamma'/\Gamma'' \cong Z_3$, and

$$\Delta(\Gamma'') = 3\Delta(\Gamma') = 3.4\pi\left(\tfrac{1}{6} - \frac{1}{2m}\right). \qquad \ldots(3.1)$$

We observe that neither x_1 nor y_1 belongs to Γ'', but x_1y_1 surely belongs to Γ''. If g is the genus of Γ'', then Γ'' is generated by $2g$ elements of infinite order and some, say s, elements of order m. We then have from (3.1)

$$12\pi\left(\tfrac{1}{6} - \frac{1}{2m}\right) = 2\pi\left(2g - 2 + \frac{m-1}{m} \cdot s\right)$$

i.e. $$3(m-1) = (m-1)\,s + 2gm. \qquad \ldots(3.2)$$

The diophantine equation (3.2) has the only solution $s = 3$, $g = 0$. Hence $\Gamma'' = (m, m, m)$. Let Γ'' be generated by x_2, y_2 satisfying

$$x_2^m = y_2^m = (x_2y_2)^m = 1.$$

Then Γ''/Γ''' is generated by u_2, v_2 satisfying

$$u_2^m = v_2^m = (u_2v_2)^m = 1, \quad u_2v_2 = v_2u_2$$

which imply $\Gamma''/\Gamma''' \cong Z_m \oplus Z_m$. Since $\Gamma'' = (m, m, m)$ satisfies the L.C.M. condition (see Theorem 1, Maclachlan 1965), its derived group Γ''' is a surface group. Now

$$\Delta(\Gamma''') = m^2\Delta(\Gamma''). \qquad \ldots(3.3)$$

Let g be the genus of Γ'''. Then the surface group Γ''' is generated by $2g$ elements of infinite order. Now (3.3) gives

$$2\pi(2g-2) = m^2 \,.\, 12\pi\left(\tfrac{1}{6} - \frac{1}{2m}\right)$$

i.e. $$g = \tfrac{1}{2}(m-1)(m-2).$$

Thus Γ''' is a surface group of genus $g = \frac{1}{2}(m-1)(m-2)$, and then $\Gamma'''/\Gamma^{\text{iv}}$ is a free abelian group of rank $2g = (m-1)(m-2)$.

Lemma 3.2 — Let G be a surface group of genus g. Then for each positive integer n, G has a characteristic subgroup G_n containing G' such that $[G : G_n] = n^{2g}$.

PROOF : Let K_n be the subgroup of G generated by the nth powers of elements of G. If α is any automorphism of G, then

$$\alpha(x_1^{nr_1} \ldots x_k^{nr_k}) = \{\alpha(x_1)\}^{nr_1} \ldots \{\alpha(x_k)\}^{nr_k},\ x_i \in G$$

where the r_i's are integers. This shows that $\alpha(K_n) \subseteq K_n$, i.e. K_n is characteristic, and therefore normal, in G. Let us set $G_n = K_nG'$. Since K_n is normal in G, $K_nG' = G'K_n$ and so G_n is a subgroup of G. G' is characteristic in G, and hence G_n is characteristic in G.

Let G have a presentation

$$\langle a_1, b_1, \ldots, a_g, b_g : \prod_{i=1}^{g} [a_i, b_i] = 1 \rangle.$$

Since a characteristic subgroup is normal, G_n is normal in G. Moreover since $G_n \supseteq G'$, G/G_n is abelian, and hence it is generated by $2g$ elements of order n which commute with each other so that

$$G/G_n \cong Z_n \oplus Z_n \oplus \ldots \oplus Z_n \text{ (}2g\text{ summands)}$$

i.e. $$[G : G_n] = n^{2g}.$$

Theorem 3.1 — Let m be an integer such that $m \geqslant 4$ and $3 \nmid m$. Then for every positive integer n, $(2, 3, 2m)$ has a solvable surface-kernel factor group of order $6m^2n^{2g}$ where $g = \frac{1}{2}(m - 1)(m - 2)$.

PROOF: Let $\Gamma = (2, 3, 2m)$. By Lemma 3.1, $|\Gamma/\Gamma'''| = 6m^2$ and Γ''' is a surface group of genus $g = \frac{1}{2}(m - 1)(m - 2)$. By Lemma 3.2, Γ''' has a characteristic subgroup $\Gamma'''_n \supseteq \Gamma^{\text{iv}}$ for each positive integer n such that $[\Gamma''' : \Gamma'''_n] = n^{2g}$. Since Γ'''_n is characteristic in Γ''', it is normal in Γ. Moreover Γ'''_n, being a subgroup of a surface group, is a surface group. Thus Γ/Γ'''_n is a solvable surface-kernel factor group of Γ and its order is $6m^2n^{2g}$.

In the above theorem, if we set $G_n = \Gamma/\Gamma'''_n$, then

$$G_n > G'_n > G''_n > G'''_n < G^{\text{iv}}_n = \{1\},$$

G'''_n being equal to G^{iv}_n only when $n = 1$. Now G_n is an automorphism group of a compact Riemann surface of genus γ which, in view of (2.3), is given by

$$6m^2n^{2g} = \frac{4\pi(\gamma - 1)}{2\pi(-2 + 3 - \frac{1}{2} - \frac{1}{3} - \frac{1}{2}m^{-1})}$$

i.e. $$\gamma = \frac{1}{2}m(m - 3)\, n^{2g} + 1.$$

We have thus proved the following theorem:

Theorem 3.2 — Let m be an integer such that $m \geqslant 4$, $3 \nmid m$ and $g = \frac{1}{2}(m - 1)(m - 2)$. Then for each positive integer n, there is a compact Riemann surface of genus $\frac{1}{2}m(m-3)\,n^{2g}+1$ which admits a solvable automorphism group G of order $6m^2n^{2g}$ such that $G>G'>G''>G'''>G^{\text{iv}}=\{1\}$, G''' being equal to G^{iv} only when $n=1$.

Putting $m=4$ in the above theorem we get Chetiya's result (1971) mentioned at the end of §1.

Acknowledgement

The author offers thanks to the referee for his kind suggestions which improved the paper.

References

Burnside, W. (1955). Theory of Groups of Finite Order. Dover Publications, Inc., New York.

Chetiya, B. P. (1971). Groups of automorphisms of compact Riemann surfaces. Ph.D. thesis, Birmingham University (U.K.).

Greenberg, L. (1960). Conformal transformations of Riemann Surfaces. *Am. J. Math.*, **82**, 749–60.

Hurwitz, A. (1892). Uber algebraische Gebilde mit eindeutigen transformationen in sich. *Math. Ann.*, **41**, 403–42.

Macbeath, A. M. (1961a). On a theorem of Hurwitz. *Proc. Glas. Math. Assoc.*, **5**, 90–96.

——————— (1961b). Discontinuous groups and birational transformations. *Proc. of Summer School in Geometry and Topology, Queen's College, Dundee.*

Maclachlan, C. (1965). Abelian groups of automorphisms of compact Riemann surfaces. *Proc. London. math. Soc.* (3), **15**, 699–712.