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ON LAGUERRE SERIES EXPANSIONS OF ENTIRE FUNCTIONS

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An analogue of Nehari's and Gilbert's results on the location of the singular points of some eigenfunction expansions is given in a case where the expansion converges to an entire function. It is shown that the Borel transform $\phi(w)$ of the entire function $f(z) = \sum_{n=0}^{\infty} a_n L_n(z)$, where $L_n(z)$ is Laguerre polynomial of degree n, has a singular point at $w = \frac{1}{1-\gamma}$ if and only if the associated power series $g(t) = \sum_{n=0}^{\infty} a_n t^n$ has one at $t = \gamma$.

Eigenfunction expansions are sometimes useful in studying the properties of the functions to which they converge. In the case where the expansion converges to an analytic function one would like to locate the singular points of this function.

Generalizing a theorem by Szegö (1954), Nehari (1956) devised a method for locating the singular points of an analytic function $f(z)$ given by a Legendre series expansion; $f(z) = \sum_{n=0}^{\infty} a_n P_n(z)$ with $\operatorname{limp\,sup}_{n\to\infty} | a_n |^{1/n} = \rho < 1$ and $P_n(z)$ is Legendre polynomial of degree n. The main idea of the proof is to relate the singularities of $f(z)$ to those of the associated power series $g(z) = \sum_{n=0}^{\infty} a_n Z^n$. This is done by finding two integral operators one that maps $f(z)$ into $g(z)$ and the other maps $g(z)$ into $f(z)$, and then applying Hadamard multiplication of singularities argument. This result was generalized later by Gilbert (1964) to ultraspherical expansions and by Gilbert and Howard (1966) to series expansions of the form $\sum_{n=0}^{\infty} a_n V_n(z)$ where $\limsup_{n\to\infty}$

$$| a_n |^{1/n} = \rho < 1$$

and $\{V_n(z)\}_{n=0}^{\infty}$ are normalized eigenfunctions of certain Sturm-Liouville system. In all the above mentioned cases both the series of eigenfunctions and the associated power series define a pair of analytic function elements in some neighbourhoods of the origin called the initial domains of definition. In all the cases under consideration, the initial domains are proper subsets of the finite complex plane. Speaking more rigorously, the initial domain of definition of the function element $g(z)$ defined by the power series is a disk with centre at the origin and radius $\rho^{-1} > 1$ while that of the function element $f(z)$ defined by the eigenfunction expansion is an ellipse whose foci are at the points ± 1 and the sum of its semi-axes is equal to ρ^{-1}. These initial domains of definition are extended by using Hadamard argument to continue the function elements analytically. Through the process of the analytic continuation, the relationship between the singularities of $f(z)$ and $g(z)$ is established.

However, in some other cases it can possibly happen that the initial domain of definition of the associated power series $g(z) = \sum_{n=0}^{\infty} a_n Z^n$ is the disk $\{Z : |Z| < \rho^{-1}; 0 < \rho < 1\}$ while that of function $f(z)$ defined by the eigenfunction expansion $f(z) = \sum_{n=0}^{\infty} a_n V_n(z)$ is the entire finite complex-plane. In other words, $f(z)$ is an entire function. Indeed, this is the case if $\{V_n(z)\}_{n=0}^{\infty}$ are Laguerre polynomials. In this case also, one would like to have results similar to those obtained earlier by Nehari and Gilbert. Of course, the singular points of $f(z)$ and those of $g(z)$ can no longer be related in a similar fashion since $f(z)$ has no finite singular point.

The main purpose of this note is to interpret the analogue of Nehari's and Gilbert's results in a case where the eigenfunction expansion converges to an entire function. It turns out that the relationship between $g(z)$ and $f(z)$ in their results is replaced by a similar one between $g(z)$ and the Borel transform $\Phi(z)$ of $f(z)$.

More precisely, we shall show that the function $f(z)$ defined by the Laguerre series expansion $f(z) = \sum_{n=0}^{\infty} a_n L_n(z)$ where $\limsup_{n\to\infty} |a_n|^{1/n} < 1$ and $L_n(z)$ is Laguerre polynomial of degree n, is an entire function of exponential type and hence its Borel transform $\Phi(z)$ is well defined. Then we show that the singularities of $\Phi(z)$ and $g(z)$ can be related in a way similar to the way Nehari and Gilbert related the singularities of $f(z)$ and $g(z)$.

Another way to interpret our results is this: we are trying to relate the behaviour at infinity of the entire function given by Laguerre series expansion to the singularities of the associated power series. This is so since the behaviour at infinity of an entire function is closely related to its indicator diagram which is (for function of exponential type) just the reflection in the real axis of its conjugate diagram (See Levin 1964). Recall that the conjugate diagram of $f(z)$ is the smallest convex domain containing all the singularities of the Borel transform $\Phi(z)$ of $f(z)$.

Let us now proceed to prove our results.

Lemma 1 — Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers, and let

$$\tau = -\overline{\lim_{n\to\infty}} \frac{\log |a_n|}{2\sqrt{n}},$$

then the series $\sum\limits_{n=0}^{\infty} a_n L_n(z)$ converges absolutely and uniformly in the interior of the parabola $\mathrm{Re}\{(-z)^{1/2}\} = \tau$, where $(-z)^{1/2}$ is taken real and positive for $z < 0$.

PROOF : We use the estimate (c.f. Szegö 1959).

$$\frac{1}{\sqrt{n}} \log |L_n(z)| \twoheadrightarrow 2\,\mathrm{Re}\{(-z)^{1/2}\}. \quad \ldots(1)$$

From the definition of τ, it follows that for every $\epsilon > 0$ we have

$$|a_n| \leqslant M(\epsilon) \exp(-(\tau - \epsilon)\, 2\sqrt{n})$$

for all but finitely many n's. Let $|\mathrm{Re}\{(-z)^{1/2}\}| \leqslant \beta < \tau$, and $\alpha = \frac{\beta + \tau}{2}$.

Then the series $A^2(\beta) = \sum\limits_{n=0}^{\infty} |a_n|^2 \exp(4\alpha\sqrt{n})$ converges for $\alpha + \epsilon < \tau$.

Hence, by Cauchy's inequality and relation (1) we obtain

$$\left|\sum_{n=0}^{\infty} a_n L_n(z)\right| \leqslant \sum_{n=0}^{\infty} |a_n L_n(z)| \leqslant \left(\sum_{n=0}^{\infty} |a_n|^2 \exp 4\alpha\sqrt{n}\right)^{1/2}$$

$$\times \left(\sum_{n=0}^{\infty} |L_n(z)|^2 \exp -4\alpha\sqrt{n}\right)^{1/2}$$

$$\leqslant A(\beta) \left(\sum_{n=0}^{\infty} \exp 4\sqrt{n}\,\mathrm{Re}\{(-z)^{1/2}\} \exp -4\alpha\sqrt{n}\right)^{1/2}$$

$$< \infty$$

for $|\mathrm{Re}\{(-z)^{1/2}\}| < \alpha$.

Q.E.D.

Corollary — If $\overline{\lim\limits_{n\to\infty}} |a_n|^{1/n} < 1$, then $f(z) = \sum\limits_{n=0}^{\infty} a_n L_n(z)$ is an entire function.

PROOF : This condition implies that $\tau = \infty$ in the previous lemma.

Lemma 2 — Let $r = \overline{\lim_{n\to\infty}} \mid a_n \mid^{1/n} < 1$, and $f(z) = \sum_{n=0}^{\infty} a_n L_n(z)$. Then $f(z)$ is an entire function of exponential type and $f(z)$ is related to the associated power series $g(t) = \sum_{n=0}^{\infty} a_n t^n$ by

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma} e^{-z/(t-1)} \frac{g(t)}{t-1}\, dt$$

where Γ is any circular path with centre at the origin and radius ρ so that $1 < \rho < 1/r$.

Proof : Consider

$$f(z) = \sum_{n=0}^{\infty} a_n L_n(z) = \sum_{n=0}^{\infty} L_n(z) \sum_{k=0}^{\infty} a_k \delta_{kn}$$

$$= \sum_{n=0}^{\infty} L_n(z) \sum_{k=0}^{\infty} a_k \frac{1}{2\pi i}\int_{\Gamma} \frac{t^{k-n}}{t}\, dt$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty} L_n(z) \int_{\Gamma} t^{-n} \sum_{k=0}^{\infty} a_k t^k \frac{dt}{t}$$

$$= \frac{1}{2\pi i}\int_{\Gamma} \sum_{n=0}^{\infty} L_n(z)\, t^{-n} \frac{g(t)}{t}\, dt .$$

Now we use the generating function

$$(1-t)^{-1} e^{-zt/(1-t)} = \sum_{n=0}^{\infty} L_n(z)\, t^n, \mid t \mid < 1 \qquad \ldots(2)$$

to get

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma} \left(1 - \frac{1}{t}\right)^{-1} e^{-z/(t-1)} \frac{g(t)}{t}\, dt$$

$$= \frac{1}{2\pi i}\int_{\Gamma} e^{-z/(t-1)} \frac{g(t)}{t-1}\, dt. \qquad \ldots(3)$$

Interchanging the integral and the summations is possible since the two series converge uniformly on Γ. If $\frac{1}{A} = \rho - 1 = \epsilon > 0$, and

$\max_{t \in \Gamma} | g(t) | = M$, the $| f(z) | < MA\rho \, e^{A|Z|} = C \, e^{A|Z|}$.

Hence, $f(z)$ is of exponential type.

Theorem 1 — Let $\{a_n\}_{n=0}^{\infty}$ and $f(z)$ be given as in Lemma 2. Then the Borel transform $\Phi(w)$ of $f(z)$ has a singular point at $w = \frac{1}{1-\gamma}$ if and only if $g(t)$ has one $t = \gamma$.

PROOF : The Borel transform of $f(z)$ (See Levin 1964) is given by

$$\Phi(w) = \int_0^{\infty} e^{-wz} f(z) \, dz \qquad \ldots(4)$$

where the integration is along the ray $z = re^{-i\theta}$, $r > 0$. Combining eqns. (3) and (4) yields

$$\Phi(w) = \int_0^{\infty} e^{-wz} f(z) \, dz = \int_0^{\infty} e^{-wz} \int_{\Gamma} \frac{1}{2\pi i} e^{-z/(t-1)} \frac{g(t)}{(t-1)} \, dt \, dz$$

$$= \frac{1}{2\pi i} \int_{\Gamma} \frac{g(t)}{t-1} \int_0^{\infty} \exp\left(-z\left(w + \frac{1}{t-1}\right)\right) dz \, dt$$

$$= \frac{1}{2\pi i} \int_{\Gamma} \frac{g(t)}{w(t-1)+1} \, dt. \qquad \ldots(5)$$

Interchanging the integrals is possible for all w such that $\operatorname{Re} w > \operatorname{Re} \frac{-1}{t-1}$ for all $t \in \Gamma$. Then, $\Phi(w)$ can be continued analytically by using the integral representation (5) to all w such that $\left| 1 - \frac{1}{w} \right| < | t |$ for all $t \in \Gamma$, since the singularities from $\frac{1}{w(t-1)+1}$ remain inside Γ. Moreover $\Phi(w)$ can be extended to points w for which $w(t-1) + 1 = 0$ for some $t \in \Gamma$. This can be done by using Hadamard's argument which goes like this. Suppose for some w_0, $w_0(t-1) + 1 = 0$ for some $t \in \Gamma$. Then we deform the contour in such a way that the new contour $\tilde{\Gamma}$ remains inside the domain of analyticity of g and $t = 1 - \frac{1}{w_0}$ lies inside $\tilde{\Gamma}$. This is always possible as long as $g(t)$ does not have a singular point at $t = 1 - \frac{1}{w_0}$. Thus, the only candidate for a singular point of $\Phi(w)$ is a common singular point of $g(t)$ and $\frac{1}{w(t-1)+1}$, i.e., if $g(t)$ has a singular point at $t = \gamma$, then the only possible singularity of $\Phi(w)$ is at $\frac{1}{1-\gamma}$.

Now we try the same procedure in the other direction. That is we find an integral operator that maps $\Phi(w)$ into $g(z)$. To do this we use the orthogonality property of Laguerre polynomials i.e.

$$\int_0^\infty L_n(x)\, L_k(x)\, e^{-x}\, dx = \delta_{nk}. \qquad \ldots(6)$$

Thus,

$$\begin{aligned} g(t) &= \sum_{n=0}^\infty a_n t^n = \sum_{n=0}^\infty t^n \sum_{k=0}^\infty a_k \delta_{kn} \\ &= \sum_{n=0}^\infty t^n \sum_{k=0}^\infty a_k \int_0^\infty L_n(x)\, L_k(x)\, e^{-x}\, dx \\ &= \int_0^\infty \sum_{n=0}^\infty L_n(x)\, t^n \sum_{k=0}^\infty a_k L_k(x)\, e^{-x}\, dx \\ &= \int_0^\infty \sum_{n=0}^\infty L_n(x)\, t^n f(x)\, e^{-x}\, dx. \end{aligned} \qquad \ldots(7)$$

Substituting (2) into (7) gives

$$\begin{aligned} g(t) &= \int_0^\infty (1-t)^{-1} \exp\left(\frac{-xt}{1-t}\right) f(x)\, e^{-x}\, dx \\ &= \int_0^\infty (1-t)^{-1} \exp\left(-x\left[\frac{1}{1-t}\right]\right) f(x)\, dx; \qquad |t| < 1. \end{aligned} \qquad \ldots(8)$$

Interchanging the integral and the summations is valid for $|t| < 1$ since both series converge uniformly for $x \in [0, \infty)$. But $f(x)$ is represented in terms of its Borel transform by the formula

$$f(x) = \frac{1}{2\pi i} \int_c e^{xw}\, \Phi(w)\, dw \qquad \ldots(9)$$

where c is any contour enclosing the conjugate diagram of f. Hence, by using eqns. (8) and (9) we obtain

$$g(t) = \frac{1}{2\pi i}\int_0^\infty (1-t)^{-1}\exp\left(-\frac{x}{1-t}\right)\int_c e^{xw}\,\Phi(w)\,dw\,dx$$

$$= \frac{1}{2\pi i}\int_c \Phi(w)(1-t)^{-1}\int_0^\infty \exp\left(-x\left[\frac{1}{1-t}-w\right]\right)dx\,dw$$

$$= \frac{1}{2\pi i}\int_c \frac{\Phi(w)}{w(t-1)+1}\,dw; \qquad \operatorname{Re}\frac{1}{1-t} > \operatorname{Re} w.$$

Interchanging the integrals is possible for the indicated values of t and w by Fubini's theorem.

Using the same reasoning as in the first part of the proof we easily see that if $\Phi(w)$ has a singular point at $w = \alpha$, then the only possible singularity of $g(t)$ is at $t = 1 - \frac{1}{\alpha}$. In other words, if $\Phi(w)$ has a singularity at $w = \frac{1}{1-\gamma}$, then $g(t)$ can have one only at $t = \gamma$ and this finishes the proof.

Q. E. D.

Example: If $a_n = \frac{\alpha^n}{(1+\alpha)^{n+1}}$; $\alpha > -\frac{1}{2}$, then it is easy to see that

$$f(x) = \sum_{n=0}^{\infty} \frac{\alpha^n}{(1+\alpha)^{n+1}} L_n(x) = e^{-\alpha x};\ 0 \leqslant x < \infty$$

$$\Phi(w) = \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{w^{n+1}} = \frac{1}{w+\alpha}$$

and $$g(z) = \sum_{n=0}^{\infty} \frac{\alpha^n}{(1+\alpha)^{n+1}} Z^n = \frac{1}{(\alpha+1)-\alpha Z}.$$

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