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INTEGRAL REPRESENTATION OF SUPERHARMONIC FUNCTIONS OF FINITE ORDER IN R^m

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This paper deals with the integral representation of superharmonic functions of order $\leqslant \lambda$ in R^m, as the sum of an integral together with a harmonic function using the kernel

$$K(x, \xi) = \begin{cases} \log \mid x - \xi \mid \text{ for } m = 2 \\ 1/ \mid x - \xi \mid^{m-2} \text{ for } m \geqslant 3 \end{cases}$$

of Hayman (1976) with the aid of Choquet's theorem in which the extreme points which support the measure of the integral representation are given by the minimal harmonic functions.

1. INTRODUCTION

In Hayman's book (1976) the integral representation of a superharmonic function $v(x)$ is given by Hadamard's theorem as

$$v(x) = u(x) - \int_{|\xi| < 1} K(x - \xi)\, d\mu(e_\xi) - \int_{|\xi| \geqslant 1} K_q(x, \xi)\, d\mu(e_\xi)$$

for a Borel measure μ in R^m, where $n(t)$ denotes the measure of the set $i \leqslant x \leqslant t$ and

$N(r) = d_m \int_0^t \frac{n(t)\, dt}{t^{m-1}}$ with $d_2 = 1$ and $d_m = m - 2$, by considering the kernel

$$K(x, \xi) = \begin{cases} \log \mid x - \xi \mid \text{ for } m = 2 \\ \dfrac{1}{\mid x - \xi \mid^{m-2}} \text{ for } m > 2. \end{cases}$$

It is proved that the order of $u(x)$ cannot exceed that of $v(x)$ and that $u(x)$ has atmost order $(q + 1)$.

Mme Hervé (1962) showed the integral representation of a positive superharmonic function as the sum of an integral and a harmonic function. In this paper an integral representation of superharmonic functions of order $\leqslant \lambda$ as the sum of an inregral and a harmonic function is attempted using the kernel $K_q(x, \xi)$ of Hayman with the aid of

Choquet's theorem. The cone of superharmonic functions of order $\leqslant \lambda$ is proved to have a metrizable, compact base and the extreme points are given by $K_{qm}(x, \xi)$ for $|\xi| \geqslant 1$, which support the measure of the integral representation.

2. TOPOLOGY OF THE CONE S (ORDER $\leqslant \lambda$)

In this section it is proved that the superharmonic functions of order $\leqslant \lambda$ form a cone denoted by S (order $\leqslant \lambda$). Consider the locally convex vector space formed of differences of superharmonic functions of order $\leqslant \lambda$. A topology induced by semi-norms on S (order $\leqslant \lambda$) is defined which is locally convex, Hausdorff and compatible with the specific order.

Definition — If $B(r)$ is a non-constant increasing function of r, for $r \geqslant r_0$, the order λ and the lower order λ_0 of $B(r)$ are defined by

$$\lambda = \limsup_{r\to\infty} \frac{\log B(r)}{\log r}$$

$$\lambda_0 = \liminf_{r\to\infty} \frac{\log B(r)}{\log r}.$$

If u is a superharmonic functions in R^2 define the order of u as the order of $B(r)$, where $B(r) = \max_{|x|=r} [-u(x)]$.

If u_1 and u_2 are two subharmonic functions in R^m of orders $\lambda(u_1)$ and $\lambda(u_2)$ respectively, then $-u_1$ and $-u_2$ are superharmonic functions of orders $\lambda(-u_1)$ and $\lambda(-u_2)$ respectively.

S (order $\leqslant \lambda$) Forms a Cone

(i) $\lambda(-u_1) + \lambda(-u_2) \leqslant \max(\lambda(-u_1), (\lambda(-u_2))$.

If $-u_1$, $-u_2$ are superharmonic functions their sum is a superharmonic function. Let $\lambda(-u_1) \geqslant \lambda(-u_2)$.

$$B(r, (-u_1)) \leqslant r^{\lambda(-u_1)+\epsilon}, \quad B(r, (-u_2)) \leqslant r^{\lambda(-u_2)+\epsilon} \leqslant r^{\lambda(-u_1)+\epsilon}$$

hence $$B(r, (-u_1) + (-u_2)) \leqslant 2r^{\lambda(-u_1)+\epsilon}$$

which implies $$\limsup_{r\to\infty} \frac{\log B(r, (-u_1) + (-u_2))}{\log r} \leqslant \lambda(-u_1) + \epsilon.$$

Since ϵ is arbitrary we get $\lambda((-u_1) + (-u_2)) \leqslant \lambda(-u_1) = \max(\lambda(-u_1), \lambda(-u_2))$.

(ii) When $c \geqslant 0$, $B(r, -cu_1) = cB(r, -u_1)$

hence $$\limsup_{r\to\infty} \frac{\log B(r, -cu_1)}{\log r} = \limsup_{r\to\infty} \frac{\log B(r, -u_1),}{\log r}$$

i.e. $$\lambda(-cu_1) = \lambda(-u_1).$$

(iii) Let $-u_1 \geqslant -u_2$ then $\lambda(-u_1) \geqslant \lambda(-u_2)$.

Hence superharmonic functions of finite order $\leqslant \lambda$ form a cone, S (order $\leqslant \lambda$).

Topology induced by seminorms on S(order $\leqslant \lambda$).
Given $v \in S$(order $\leqslant \lambda$) associate to each point $x \in R^m$, the Riesz measure (harmonic measure) ρ_x^ω for an open set ω of x in R^m. Consider pairs (u, v) of superharmonic functions of order $\leqslant \lambda$ on an open set ω of S(order $\leqslant \lambda$). Define an equivalence relation in the set of pairs (u, v) denoted by $(u_1, v_1) \sim (u_2, v_2)$ if $u_1 + v_2 = u_2 + v_1$ or $u_1 - u_2 = v_1 - v_2$. These equivalence classes form a vector space over the real number field. The equivalence class containing (u, v) is denoted by $[u, v]$. i.e. $c\,[u, v] = [cu, cv]$; $[u_1, v_1] + [u_2, v_2] = [u_1 + u_2, v_1 + v_2]$. The natural order on S(order $\leqslant \lambda$) denoted by $[u_1, v_1] \geqslant [u_2, v_2]$ is $(u_1 - v_1) \geqslant (u_2 - v_2)$. The specific order ($\succ$) is defined by

$$[u_1, v_1] \succ [u_2, v_2] \text{ iff } [u_1, v_1] - [u_2, v_2] = [w, 0] \text{ for some } w \geqslant 0, \text{ or}$$
$$u_1 - v_1 = u_2 - v_2 + w.$$

Since order of $c[u, v]$ with respect to superharmonic functions gives

$$\lambda(c[u, v]) = \lambda[u, v] \text{ also } \lambda\{[u_1, v_1] + [u_2, v_2]\} = -\max\,(\lambda\,[u_1, v_1], \lambda[u_2, v_2])$$

the order of the scalar product and sum is hence $\leqslant \lambda$. Moreover, the set of superharmonic functions is a complete lattice for the specific order.

Theorem 1 — For the elements $[u, v]$ of S (order $\leqslant \lambda$), $|\int u d\rho_x^\omega - \int v d\rho_x^\omega|$ for a fixed regular domain ω and a fixed $x \in \omega$ is a seminorm. All these seminorms define on S (order $\leqslant \lambda$) a topology T which is locally convex, Hausdorff and compatible with the specific order.

$$|\int ud\,\rho_x^\omega - \int vd\,\rho_x^\omega| = |\int (u - v)\, d\rho_x^\omega| \text{ is a seminorm (Schaeffer 1969),}$$

since

(i) $$|\int\{(u - v) + (u_1 - v_1)\}|\, d\rho_x^\omega = |\int (u - v)\, d\rho_x^\omega + \int (u_1 - v_1)\, d\rho_x^\omega|$$
$$\leqslant |\int (u - v)\, d\rho_x^\omega| + |\int (u_1 - v_1)\, d\rho_x^\omega|;$$

(ii) $$|\int \alpha\,(u - v)\, d\rho_x^\omega| = |\alpha|\, |\int (u - v)\, d\rho_x^\omega|;$$

(iii) $|\int (u - v)\, d\rho_x^\omega| = 0$ does not always imply $(u - v) = 0$.

Also differences of superharmonic functions, functions of order $\leqslant \lambda$ is linear from (i) and (ii), therefore forms a vector space.

Local convexity — $A \subset R^m$ is convex if

$$(x, y) \in A \Rightarrow cx + (1 - c)\, y \in A \text{ for } c \leqslant 1.$$

i.e. to prove $(u - v, u_1 - v_1) \in A \Rightarrow c(u - v) + (1 - c)(u_1 - v_1) \in A$

$$\begin{aligned} c(u - v) + (1 - c)(u_1 - v_1) &= (u_1 - v_1) + c(u - v) - c(u_1 - v_1) \\ &= (u_1 - v_1) + c(u - u_1) - c(v - v_1) \\ &= (u_1 - v_1) \in A \end{aligned}$$

since $(u - u_1) = (v - v_1)$ from equivalence class definition. Hence the vector space formed of differences of superharmonic functions of order $\leqslant \lambda$ is locally convex.

The topology is separated.

Let the seminorms be zero i.e. $u\rho_x^{\omega} = v\rho_x^{\omega}$ then to prove that $u(x) = v(x)$, since $u\rho_x^{\omega} = v\rho_x^{\omega}$ the superharmonic functions u and v coincide at x in ω, i.e. $u(x) = v(x)$.

Definition 2 — T is the least fine topology rendering continuous for each point $x \in R^m$ the linear mapping $(u - v) \to \int (u - v)\, d\rho_x^{\omega}$ of S in the space of measures on R^m provided with the vague topology. T is therefore a topology of the locally convex space defined by the seminorms $n : (u, v) \to | \int (u - v)\, d\rho_x^{\omega} |$.

Neighbourhood Base for the Locally Convex Topology

Since the arbitrary family of seminorms $\{n_\alpha : \alpha \in A\}$ determine a locally convex topology, for each $\alpha \in A$, let $U_\alpha = \{(u, v) \in S \text{ (order} \leqslant \lambda), n_\alpha(u, v) \leqslant 1\}$

$n_\alpha(u, v) = | \int (u - v)_\alpha \, d\rho_x^{\omega} |$ restriction of $(u - v)$ to the convex subset α, hence order is preserved.

$U = \cap \{U_\alpha\}$ = finite intersections of U_α, then the family $n^{-1}(U)$ where U ranges over all finite intersections U_α is a 0 neighbourhood base for the locally convex topology.

3. Metrizability of the Base of the Cone S(order $\leqslant \lambda$)

For the locally convex vector space E'let S_{x_0, ω_0} of superharmonic functions of order $\leqslant \lambda$ satisfy the conditions $\int v\, d\rho_{x_0}^{\omega_0} = 1$ for a regular domain $\omega_0 \in E'$, $x_0 \in \omega_0$. It is a base of S. Since the cone consists of superharmonic functions of order $\leqslant \lambda$ intersection with a hyperplane $\int v\, d\rho_{x_0}^{\omega_0}$ gives a base in which the order of the superharmonic functions is $\leqslant \lambda$. It is essential to prove that the base is metrizable and compact in order to use Choquet's theorem for representation of superharmonic functions as an integral with respect to a measure supported by the extreme points plus a harmonic function.

Since the topology T is separated if it can be defined on S (order $\leqslant \lambda$) by a countable family of deviations, it is metrizable (Brelot 1967). Let $\varkappa$ be a countable base dense in E' ! The seminorm is given by $(u, v) \to |\int (u - v)\, d\rho_x^{\omega}|$. The deviation is defined by $(u, v) \to E_p(u, v) = |u(x_p) - v(x_p)|$. Given a T neighbourhood of $v_0 \in S$ (order $\leqslant \lambda$) defined by $E(u, v_0) = |u(x) - v_0(x)| \leqslant \alpha$ for $x \in R^m$, $\alpha > 0$. It contains a T neighbourhood of v_0 of the form

$$E_{p_k}(u, v_0) \leqslant \alpha_k,\ k = 1, 2, \ldots N,\ \alpha_k \leqslant \alpha.$$

Now, given a superharmonic function there exists a sequence of superharmonic functions converging to u, u is contained in a neighbourhood U_α. Let U_{α_k} be a compact neighbourhood of U_α with $\bar{U}_{\alpha_k} \subset E'$, then there is an increasing sequence of superharmonic functions U_k on U_{α_k} such that $u = \lim_{k\to\infty} u_k$ on U_{α_k}.

Hence $\quad E_{p_k}(u_k, v_0) \leqslant \alpha_k,\ k = 1, 2, \ldots n$

i.e. $\quad u_k(x_p) - v_0(x_p) \leqslant \alpha_k.$

$$u_k(x_p) \leqslant v_0(x_p) + \alpha_k \leqslant v_0(x_p) + \alpha/4, \text{ i.e., } u(x_p) \leqslant v_0(x_p) + \alpha.$$

Determination of x_p : Let X be a domain containing x. Given $\epsilon > 0$ there exists a neighbourhood of $x \in X$ such that

$$\left.\begin{array}{l}\xi, \xi' \leqslant \delta \\ u \text{ harmonic} \geqslant 0 \text{ in } X\end{array}\right\} \Rightarrow \frac{u(\xi')}{1+\epsilon} \leqslant u(\xi) \leqslant (1+\epsilon)\, u(\xi')$$

Choose $\quad \epsilon = \dfrac{\alpha/4}{v_0(x) + \alpha}$ and $x_p \in X$.

Hence $|u(x_p) - v_0(x_p)| < \frac{1}{4}\alpha$ is the T neighbourhood sought. Hence the T topology is metrizable.

4. Compactness of the Base of the Cone S (order $\leqslant \lambda$)

Given a pair (ω_0, x_0) denote by $S_{\alpha, \omega_0, x_0}$ (order $\leqslant \lambda$) (written S_α (order $\leqslant \lambda$) the set of functions $v \in S$(order $\leqslant \lambda$) such that $\int v d\rho_{x_0}^{\omega_0} \leqslant \alpha$, $\alpha > 0$. Intersection of the cone S (order $\leqslant \lambda$) with a hyperplane $\int v d\rho_{x_0}^{\omega_0} \leqslant \alpha$ forms a base of the cone.

Lemma 2 — Given $v_n \in S_\alpha$ (order $\leqslant \lambda$) one can extract a subsequence v_{n_k} such that for each $x \in E'$, the measures $v_{n_k}\rho_x^{\omega}$ converge vaguely in the space of measure on E'.

Proof : Let x_p be a countable dense set of E'. Consider for each p the measure $v_n\rho_x^{\omega}$ on E' for a compact set $K \in E'$ the restriction to K of the measures of $v_n\rho_{x_p}^{\omega}$ have bounded masses in that set, hence $v_n \rho^*_{x_p}$ is bounded at x_p independent of n. The diagonal procedure helps to extract a subsequence n_k such that for each p, $v_{n_k} \rho^*_{x_p}$ converge vaguely. Since any superharmonic function is determined by its values outside a polar set, $v(x_p) = \lim \int v_{n_k} d\rho^*_{x_p}$ according to the filter f_x of sections of the decreasing directed set of all regular domains containing x_p, hence, for every $x_p \in E'$, $\int v_{n_k} \rho^*_{x_p}$ converges vaguely in E' towards a measure μ^x on E'.

Proposition 3 — Let v_n (order $\leqslant \lambda$) be a sequence in S (order $\leqslant \lambda$) such that for each $x \in E'$ the measures $v_n \rho^*_x$ converge vaguely in the space of measures on $\bar{E}'$ then the sequence v_n is T convergent towards $\widehat{\liminf_n}\, v_n$.

PROOF : v_n is a superharmonic function of order $\leqslant \lambda$. Hence $\widehat{\liminf_n}\, v_n$ is a superharmonic function whose order $\leqslant \lambda$. $\widehat{\text{Lim inf}_n}\, v_n$ is harmonic in $E' \cap cv_n$. By the previous lemma we can extract a subsequence $\int v_{n_k} \rho^*_x$ of $\int v_n \rho^*_x$ which converges vaguely in the space of measures on E' to $\widehat{\liminf_n}\, v_n$.

Lemma 4 — The sequence v_n is T convergent in S^+ (order $\leqslant \lambda$). Consider the pair (ω_0, x_0)

(1) $\int v_n \, d\rho^{*0}_{x_0}$ is bounded independent of n.

(2) If the T limit of v_n is zero, $\int v_n \, d\rho^{*0}_{x_0} \to 0$.

PROOF : Let $c \in C^+(\bar{E}')$, $0 \leqslant c \leqslant 1$. S_c and S_{1-c} are 2 disjoint elements of $\bar{E}'$. We can choose 2 pairs (ω_1, x_1) and (ω_2, x_2) such that $\bar{\omega}_1 \subset E' \cap S_c$, $\bar{\omega}_2 \subset E' \cap S_{1-c}$.

We have $v_n = (v_n)_c + (v_n)_{1-c}$.

$$\int (v_n)_c \, d\rho^{*1}_{x_1} = (v_n)_c(x_1) \quad \text{and} \quad \int (v_n)_{1-c} \, d\rho^{*2}_{x_2} = (v_n)_{1-c}(x_2).$$

are bounded independent of n. Therefore also

$$\int (v_n)_c \, d\rho^{*0}_{x_1} \qquad \text{and} \qquad \int (v_n)_{1-c} \, d\rho^{*0}_{x_0}$$

(ω_0, x_0), (ω_1, x_1) are contained in $\bar{E}'$ by axiom 3 there exists a number k such that

$$1/k \leqslant \frac{\int (v_n)_c \, d\rho^{*1}_{x_1}}{\int (v_n)_c \, d\rho^{*0}_{x_0}} \leqslant k.$$

Therefore $\int (v_n)\, d\rho_{x_0}^{\omega_0}$ is bounded independent of n. Similarly (2).

Theorem 5 — Being given a pair (ω_0, x_0) and a number $\alpha > 0$, the set S(order $\leqslant \lambda$) of functions $v \in S$ (order $\leqslant \lambda$) such that $\int v\, d\rho_{x_0}^{\omega_0} \leqslant \alpha$ is compact for the topology T. (1) Therefore the cone S (order $\leqslant \lambda$) provided with the topology T is locally compact. (2) We can obtain a compact base of the cone S (order $\leqslant \lambda$) by considering the set of functions $v \in S$ (order $\leqslant \lambda$) such that $v(x_1) + v_{\alpha}(x_2) = 1$.

PROOF : (1) S_α (order $\leqslant \lambda$) is compact for the topology T. For from the sequence $v_n \in S$(order $\leqslant \lambda$) we can extract a subsequence v_{n_k} satisfying conditions of Proposition 3 such that v_{n_k} T converges towards $\widehat{\liminf_k}\, v_{n_k}$ and $\widehat{\liminf}\, v_{n_k} \in S_\alpha$(order $\leqslant \lambda$). Since S_α consists of superharmonic functions of order $\leqslant \lambda$, $\liminf v_{n_k}$ is a superharmonic function of order $\leqslant \lambda$. Also $\int \liminf v_{n_k}\, d\rho_{x_0}^{\omega_0} \leqslant \liminf \int v_{n_k}\, d\rho_{x_0}^{\omega_0} \leqslant \alpha$ as S (order $\leqslant \lambda$) is a T neighbourhood of the origin in the cone S (order $\leqslant \lambda$) which is locally compact.

(2) The set $A = v(x_1) + v(x_2) = 1$ is closed therefore compact and is contained in an S (order $\leqslant \lambda$). Consider the elements $c, \omega_1, x_1, \omega_2, x_2$ of Lemma 4, $v \in A$ implies $\int v_c\, d\rho_{x_1}^{\omega_1}$ and $\int v_{1-c}\, d\rho_{x_2}^{\omega_2}$ are bounded by fixed numbers. Therefore also $\int v d\rho_{x_0}^{\omega_0}$ is bounded. Further $v_1(x_1) + v_2(x_2) = 0$ implies $v = 0$, and the superharmonic function v has order $\leqslant \lambda$. Hence A meets every generator of the cone S(order $\leqslant \lambda$).

5. INTEGRAL REPRESENTATION OF THE ELEMENTS OF S (ORDER $\leqslant \lambda$)

Recall Choquet's fundamental results (Meyer 1966).

Choquet's Theorem — In a separated and locally compact topological vector space S, let S be a convex cone with compact base and let ϵ be the set of extreme elements of A.

Theorem I — If A is metrizable every point in S (order $\leqslant \lambda$) is the resultant of atleast a Radon measure $\geqslant 0$, supported by the extreme points. i.e.

Every point of S(order $\leqslant \lambda$) possesses atleast an integral representation of the form $u(x) = \int_{|\xi| \geqslant 1} K_q(x,\xi)\, d\mu(e_\xi) + v(x)$ where $\mu(e_\xi)$ is a Radon measure $\geqslant 0$ on A supported by ϵ.

Theorem II — If the cone S (order $\leqslant \lambda$) is a lattice for the order relation defined in S (order $\leqslant \lambda$) the preceeding integral representation is unique. Denote by E the potentials with point support or minimal harmonic functions > 0 in E'.

Definition — A harmonic function in a given domain is called minimal for this domain if it dominates there no harmonic function except its own constant submultiples.

For example, in space for unit sphere, centre O $u(P) = \int_{OS=1} F(S, P)\, d\mu(e_\xi)$ with $F(S, P) = (1 - \overline{OP}^2)/(\overline{SP})^3$ where $\mu(e)$ is a finite, nonnegative completely additive function of Borel sets.

Lemma 6 — Suppose that the superharmonic function $u(x)$ has order λ is harmonic and minimal. Let A be any Borel set of the boundary $\Delta = (\hat{E}' - E')$. If now a relation of the form

$$u(x) \leqslant \int_{|\xi| \geqslant 1} K_q(x, \xi)\, d\mu(e_\xi) \qquad \ldots(1)$$

obtains for all P in D where $q + 1 < \lambda$, $u(x) = u(x_0)\, K_q(x, \xi_0)$ where ξ_0 is some point of A.

PROOF : A has a finite order superharmonic function. Therefore has a closed subset A_1 containing a superharmonic function of order $\leqslant \lambda$, all of them having a diameter less than some selected positive number. Atleast one subset has a superharmonic function of order $\leqslant \lambda$, say A_2. By proceeding inductively it is possible to construct a decreasing sequence $A_1, A_2, \ldots A_n$ of closed subsets of A whose diameters approach one and has a superharmonic function of order $\leqslant \lambda$. Let x be a unique point common to all the A_n. Since A_n has a superharmonic function of order $\leqslant \lambda$, the integral (1) can be extended over A_n instead of A which represents a harmonic function of order $\leqslant \lambda$, dominated by the minimal function $u(x)$ and is equal to $c_n(u(x))$ where c_n is positive. If $\mu_n(e) = c_n^{-1}\, \mu(A_n.e)$ where $\mu(A_n)$ is the measure associated to the superharmonic function of order λ in A_n hence $u(x) = \int A_n K_q(x, \xi)\, d\mu_n(x)$. The total mass of the distribution $\mu_n(e)$ has a limit a point mass of amount $u(x_0)$ located at x.

Consider $K_q(x, \xi)\, d\rho^{*}_{\xi_0}$.

Lemma 7 — Any harmonic function of finite order on E' (In R^m) can be represented by $u(x) = K_q(x, \xi)\, d\mu(\xi)$ where μ is a positive measure on Δ.

PROOF : Consider an exhaustion $E_n, E_n \subset \bar{E}_n \subset E'$, $\cup E_n = E'$. $\hat{R}^u_{E_n}$ is a potential $\leqslant \lambda G\xi_0$ where λ satisfies $\lambda\, G\xi_0 \geqslant 0$ on ∂E_n

$$\hat{R}_{E_n}(x) = \int G(x, \xi)\, d\mu_n(\xi) = \int K_q(x, \xi)\, d\nu_n(\xi)$$

where $d\nu_n(x) = G(x_0, \xi_0)\, d\mu_n(\xi)$ where $d\nu_n = u(x_0)$ support of $\nu_n \subset \delta E_n$. We may extract a subsequence ν_{n_p} converging vaguely to a positive measure μ on E with its support on Δ. Continuity of $x \to K_q(x, \xi)$ on $\hat{E}$ gives $u(x) = \int K_q(x, \xi)\, d\mu(e_\xi)$.

Theorem 8 — Any minimal harmonic function of order $\leqslant \lambda$, is equal to $u(\xi_0)\,K(x, \xi)$ for some $x \in \Delta$. The corresponding X are all called minimal points of Δ and their set is denoted by Δ_1 (Brelot 1971).

PROOF : Let $u(x) = \int K(x, \xi)\, d\mu(X)$ where μ is a strictly positive measure. On Δ there exists a point ξ_0 such that any open neighbourhood U of ξ_0 in E has a measure $\neq 0$. Hence

$$\int_u K_q(x, \xi)\, d\mu(X) = cu(\xi) \text{ with } c = \frac{\mu(u)}{\mu(\xi_0)}$$

$$\frac{u(\xi)}{K(x, \xi_0)\, u(\xi_0)} = \frac{\int_U \dfrac{K_q(x, \xi)\, d(X)}{K_q(x, \xi_0)}}{\int_U d\mu}$$

Let x be fixed. Given $\epsilon > 0$, $1 - \epsilon < \dfrac{K_q(x, \xi)}{K(x_0, \xi_0)} < 1 + \epsilon$ for $X \in U$.

If $K_q(x, \xi)\, d\mu(X)$ is minimal it is proportional to $K_q(x_0, \xi)$ where X_0 is the closed support of μ on Δ. The correspondence $X \to K_x$ of Δ on B_{x_0}, Δ compact, and B'_{x_0} of K_x on B_{x_0} is a homeomorphism because this mapping $X \to K_x$ of Δ on B'_{x_0} (Hausdorff space) is continuous and one-one. The image of Δ_1 is the set of extreme points of B_{x_0}.

Let us recall the following:

Weierstrass's Representation Theorem — Suppose that μ is a Borel measure in R^m, let $n(t)$ be the measure of $D(0, t)$ and let $q(t)$ be a positive integer valued function of t continuous on the right and so chosen that $\int_1^\infty (t_0/t)^{q(t)+m-l}\, dn(t) < \infty$ for all positive t_0. Then there exist functions $u(x)$ subharmonic in R^m and with Riesz measure μ and all such functions take the form

$$u(x) = \int_{|\xi|<1} K(x - \xi)\, d\mu e_\xi + \int_{|\xi|\geqslant 1} K_{q(|\xi|)}(x, \xi)\, d\mu e_\xi + v(x).$$

Then by Lemma 4.4 and Theorem 4.2 of Hayman (1976), if $m = 2$, $x = R > 2$

$$\int_{|\xi|<1} K(x - \xi)\, d\mu e_\xi = \int_{|\xi|<1} \log |x - \xi|\, d\mu e_\xi \geqslant 0$$

and if $\quad m > 2, \quad K(x - \xi)\, d\mu e_\xi \to 0$ as $x \to \infty$.

Thus in either case the order of $u(x)$ cannot exceed that of $v(x)$. Thus $u(x)$ has atmost order $q + 1$. Also $q + 1 \leqslant \lambda$, hence the superharmonic functions belong to S (order $\leqslant \lambda$).

Since the base of the cone S (order $\leqslant \lambda$) is metrizable and the extreme points are given by $K_q(x, \xi)$ for $|\xi| \geqslant 1$, as also the superharmonic functions of order $\leqslant \lambda$, form a complete lattice under the specific order, by the theorems of Choquet.

$$u(x) = \int_{|\xi| \geqslant 1} K_q(x, \xi)\, d\mu e_\xi + v(x).$$

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References

Brelot, M. (1967). Lectures of Potential Theory. Tata Institute of Fundamental Research, Bombay.

———— (1971). On Topologies and Boundaries in Potential Theory. Springer Verlag, Berlin.

Hayman, W. K., and Kennedy J. F. (1976). Subharmonic Functions, Vol I. Academic Press, New York.

Hervé, M. (1962). Recherches Axiomatiques sur la theorie des fonctions surharmoniques et du potentiel. *Annals Inst. de Fourier*, **12**, 451–571.

Meyer, P. A. (1966). Probability and Potentials. Blaisdell Publications, New York.

Schaeffer, H. H. (1969). Topological Vector Spaces. Springer-Verlag, Berlin.