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INTEGRALS INVOLVING SPHEROIDAL WAVE FUNCTIONS

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This paper deals with the evaluation of two definite integrals involving spheroidal wave function, the H-function of several complex variables defined by Srivastava and Panda (1976), and the generalized hypergeometric function. Also, an expansion formula for the product of the generalized hypergeometric function and the (Srivastava-Panda) H-function of several complex variables has been obtained. Since the H-function of several variables, spheroidal wave function and the generalized hypergeometric function may be expressed in terms of a number of higher transcendental functions and polynomials, the results obtained in this paper include some known results as their particular cases.

1. INTRODUCTION

The H-function of several variables has been recently defined and represented by Srivastava and Panda (1976). For convenience and brevity, we shall use the contracted notations introduced by Srivastava and Panda (1976) throughout the present paper.

The known results (Flammer 1957, p. 16; Erdélyi *et al.* 1954, p. 316; Milne-Thomson 1933, p. 33) required in the sequel may be recalled here as follows:

(i) The spheroidal wave function can be expressed as:

$$S_{mn}(c, x) = \sum_{r=0,1}^{\infty}{}^{*} \; d_r^{mn}(c)\, P_{m+r}^{m}(x) \qquad \ldots(1.1)$$

where the coefficients $d_r^{mn}(c)$ satisfy the recursion formula [Flammer 1957, eqn. (3.1.4)] and the asterisk over the summation sign indicates that the sum is taken over only even or odd values of r according as $(n - m)$ is even or odd.

(ii) $$\int_{-1}^{1} (1 - x^2)^{\rho-1} P_\nu^m(x)\, dx$$

$$= \frac{\pi\, 2^m \Gamma(\rho + \frac{1}{2}m)\, \Gamma(\rho - \frac{1}{2}m)}{\Gamma(1 + \frac{1}{2}(\nu - m))\, \Gamma(\frac{1}{2} - \frac{1}{2}(\nu + m))\, \Gamma(\rho - \frac{1}{2}\nu)\, \Gamma(1 + \rho + \frac{1}{2}\nu)} \qquad \ldots(1.2)$$

provided that $2\,\mathrm{Re}\,(\rho) > |\,\mathrm{Re}\,(m)\,|$.

$$\text{(iii)} \quad E_a f(a) = f(a+1),\ E_a^n f(a) = E_a\,[E_a^{n-1} f(a)] \qquad \ldots(1.3)$$

where E denotes the finite difference operator. Also, we shall use the following notation throughout the paper:

$$(\alpha)_r = \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} = \alpha(\alpha+1)\ldots(\alpha+r-1). \qquad \ldots(1.4)$$

2. Finite Integrals

The main integrals to be proved here are the following:

$$J_1(\rho) = \int_{-1}^{1} (1-x^2)^{\rho-1} S_{mn}(c,x)\, H[z_1(1-x^2)^{\sigma_1}, \ldots, z_r(1-x^2)^{\sigma_r}]\, dx$$

$$= 2^m \pi \sum_{r=0,1}^{*} d_r^{mn}(c)\, [\Gamma(1+\tfrac{1}{2}r)\,\Gamma(\tfrac{1}{2}-m-\tfrac{1}{2}r)]^{-1}$$

$$H^{0,\lambda+2:(\mu',\nu');\ldots;(\mu^{(r)},\nu^{(r)})}_{A+2,C+2:(B',D');\ldots;(B^{(r)},D^{(r)})}\left(\begin{matrix}[1-\rho+\frac{1}{2}m:\sigma_1,\ldots,\sigma_r]\\ [1-\rho+\frac{1}{2}(m+r):\sigma_1,\ldots,\sigma_r]\end{matrix}\right.$$

$$\begin{matrix}[1-\rho-\frac{1}{2}m:\sigma_1,\ldots,\sigma_r]:[(a):(\theta'),\ldots,\theta^{(r)}]:[(b'):\phi'];\ldots;\\ [-\rho-\frac{1}{2}(m+r):\sigma_1,\ldots,\sigma_r]:[(c):\psi',\ldots,\psi^{(r)}]:[(d'):\delta'];\ldots;\end{matrix}$$

$$\left.\begin{matrix}[(b)^{(r)}:\phi^{(r)}]\\ [(d)^{(r)}:\delta^{(r)}]\end{matrix}; Z_1,\ldots,Z_r\right) \qquad \ldots(2.1)$$

provided that σ_i are positive numbers such that

$$\Delta_i + \sigma_i > 0,\ |\arg(Z_i)| < \tfrac{1}{2}(\Delta_i+\sigma_i)\pi,$$

$$\mathrm{Re}\,(\rho) > 0 \text{ and } \mathrm{Re}\,[\rho + \sum_{i=1}^{r} \sigma_i\alpha_i] > 0,\ i = 1,\ldots,r,$$

where

$$\Delta_i = -\sum_{=\lambda+1}^{A} \theta_j^{(i)} + \sum_{j=1}^{\nu^{(i)}} \phi_j^{(i)} - \sum_{j=\nu^{(i)}+1}^{B^{(i)}} \phi_j^{(i)}$$

$$-\sum_{j=1}^{C} \psi_j^{(i)} + \sum_{j=1}^{\mu^{(i)}} \delta_j^{(i)} - \sum_{j=\mu^{(i)}+1}^{D^{(i)}} \delta_j^{(i)} > 0,\ i = 1,\ldots,r$$

$$\alpha_i = d_j^{(i)} / \delta_j^{(i)},\ j = 1,\ldots,\mu^{(i)}.$$

and the series on the right-hand side of (2.1) is assumed to be convergent.

$$J_2(\rho) = \int_{-1}^{1} (1 - x^2)^{\rho-1} S_{mn}(c, x)\, {}_uF_v \left[\begin{matrix} \xi_u \\ \eta_v \end{matrix} ; h(1 - x^2)^k \right]$$

$$H\,[z_1(1 - x^2)^{\sigma_1} ; \ldots ; z_r(1 - x^2)^{\sigma_r}\,]\, dx$$

$$= 2^m\pi \sum_{r=0,1}^{\infty}{}^{*} \sum_{p'=0}^{\infty} d_r^{mn}(c)\,[\Gamma(1 + \tfrac{1}{2}r)\, \Gamma(\tfrac{1}{2} - m - \tfrac{1}{2}r)]^{-1}$$

$$\times \frac{\prod_{j=1}^{u} (\xi_j)\,(h)^{p'}}{\prod_{j=1}^{v} (\eta_j)\, p'\,!}\; H^{0,\lambda+2\,:\,(\mu',\nu');\,\ldots\,;\,(\mu^{(r)},\nu^{(r)})}_{A+2,C+2\,:\,(B',D');\ldots\,;\,B^{(r)},D^{(r)})}$$

$$\begin{pmatrix} [(a) : \theta', \ldots, \theta^{(r)}] : (1 - \rho - p'k + \tfrac{1}{2}m : \sigma_1, \ldots, \sigma_r), \\ [(c) : \psi', \ldots, \psi^{(r)}] : (1 - \rho - p'k + \tfrac{1}{2}(m + r) : \sigma_1, \ldots, \sigma_r) \end{pmatrix.$$

$$(1 - \rho - p'k - \tfrac{1}{2}m : \sigma_1, \ldots, \sigma_r) :$$
$$(- \rho - \tfrac{1}{2}(m + r) - p'k : \sigma_1, \ldots, \sigma_r) :$$

$$\left. \begin{matrix} (b' : \phi'), \ldots, (b^{(r)}, \phi^{(r)}) \\ (d', \delta'), \ldots, (d^{(r)}; \delta^{(r)}) \end{matrix} ;\; z_1, \ldots, z_r \right) \qquad \ldots(2.2)$$

where h, k are positive integers (either h or k may be zero). σ_i are positive numbers such that $\Delta_i + \sigma_i > 0$ and

$$|\arg (z_i)| < \tfrac{1}{2}(\Delta_i + \sigma_i)\,\pi,\ \text{Re}\,(\rho) > 0,\ \text{Re}\,(\rho + \sum_{i=1}^{r} \sigma_i\alpha_i) > 0,$$

$i = 1, \ldots, r$ and Δ_i, σ_i are given in (2.1). The result (2.2) holds if $u \leqslant v (u = v + 1$ and $|h| < 1)$, none of $\eta_1, \eta_2, \ldots, \eta_v$ is zero or a negative integer, and with the remaining conditions as stated in (2.1). The series on the right-hand side of (2.2) is assumed to be convergent.

PROOF OF (2.1) : To prove (2.1), we first express the spheroidal wave function $S_{mn}(c, x)$ in the series form (1.1), and the H-function of several variables

$$H\,[z_1(1 - x^2)^{\sigma_1}, \ldots, z_r(1 - x^2)^{\sigma_r}\,]$$

in terms of multiple contour integral form as given by Srivastava and Panda [1976, p. 130, eqn. (1.3)]. Now, changing the order of integration and summation, evaluating the inner integral with the help of (1.2), and finally reinterpreting the multiple contour integral thus obtained by the definition of the H-function of several variables given by Srivastava and Panda [1976, p. 130, eqn. (1.3)], we get the desired result (2.1).

Regarding the interchange of the order of integration and summation it is observed that x-integral is convergent if

$$\text{Re}(\rho) > 0; \text{Re}(\rho + \sum_{i=1}^{r} \sigma_i \alpha_i) > 0, \quad i = 1, ..., r.$$

The multiple contour integral converges under the conditions stated in (2.1). The series

$$\sum_{r=0,1}^{\infty}{}^{*} d_r^{mn}(c) P_{m+r}^{m}(x)$$

converges absolutely and uniformly for all finite x (Flammer 1957, pp. 16-17). Hence the interchange of order of integration and summation is justified (Bromwich 1965, p. 504).

PROOF OF 2.2 : On multiplying both sides of (2.1) by

$$\prod_{j=1}^{u} \Gamma(\xi_j + \delta) (h)^{\delta} / \prod_{j=1}^{v} \Gamma(\eta_j + \delta)$$

and applying the operator $\exp(E^k E_\delta)$ yields

$$\exp(E^k E_\delta) [J_1(\rho) \prod_{j=1}^{u} \Gamma(\xi_j + \delta) h^{\delta} / \prod_{j=1}^{v} \Gamma(\eta_j + \delta)]$$

$$= 2^m \pi \exp(E^k E_\delta) \sum_{r=0,1}^{\infty}{}^{*} d_r^{mn}(c) [\Gamma(1 + \tfrac{1}{2}r) \Gamma(\tfrac{1}{2} - \tfrac{1}{2}r - m)]^{-1}$$

$$\times \frac{\prod_{j=1}^{u} \Gamma(\xi_j + \delta) h^{\delta}}{\prod_{j=1}^{v} \Gamma(\eta_j + \delta)} H^{0, \lambda+2 : (\mu', \nu'); \ldots; (\mu^{(r)}, \nu^{(r)})}_{A+2, C+2 : (B', D'); \ldots; (B^{(r)}, D^{(r)})}$$

$$\left(\begin{matrix} [(a) : \theta', \ldots, \theta^{(r)}] : (1-\rho+\tfrac{1}{2}m : \sigma_1, \ldots, \sigma_r), (1-\rho-\tfrac{1}{2}m : \sigma_1, \ldots, \sigma_r) : \\ [(c) : \psi', \ldots, (\psi^{(r)})] : (1-\rho+\tfrac{1}{2}(m+r) : \sigma_1, \ldots, \sigma_r), (-\rho-\tfrac{1}{2}(m+r); \end{matrix} \right.$$

$$\left. \begin{matrix} : (b', \phi'); \ldots; (b^{(r)} : \phi^{(r)}) \\ ; (\sigma_1, \ldots, \sigma_r) : (d', \delta'); \ldots; (d^{(r)} : \delta^{(r)}) \end{matrix} ; z_1, \ldots, z_r \right). \quad \ldots(2.3)$$

Summing both sides of (2.3) and using the definition of the finite difference operator (1.3), we get

$$\sum_{p=0}^{\infty} \left\{ \frac{\prod_{j=1}^{u} \Gamma(\xi_j + \delta + p) (h)^{\delta+p}}{\prod_{j=1}^{v} \Gamma(\eta_j + \delta + p) p!} \int_{-1}^{1} (1 - x^2)^{\rho+p^{k}-1} \right.$$

$$\left. \times S_{mn}(c, x) H [z_1(1 - x^2)^{\sigma_1}, \ldots, z_r(1 - x^2)^{\sigma_r}] dx \right\} =$$

(*equation continued on p.* 1340)

$$= 2^m \pi \sum_{p=0}^{\infty} \sum_{r=0,1}^{\infty}{}^{*} d_r^{mn}(c) \left[\Gamma(1+\tfrac{1}{2}r)\,\Gamma(\tfrac{1}{2} - \tfrac{1}{2}r - m)\right]^{-1}$$

$$\times \frac{\prod_{j=1}^{u} \Gamma(\xi_j + \delta + p)}{\prod_{j=1}^{v} \Gamma(\eta_j + \delta + p)\, p\,!} H^{0,\lambda+2\,:\,(\mu',\nu');\,\ldots;\,(\mu^{(r)},\nu^{(r)})}_{A+2,\;C+2\,:\,(B',D');\,\ldots;\,(B^{(r)},D^{(r)})}$$

$$\times \left(\begin{matrix} [(a):\theta',\ldots,\theta^{(r)}] : (1-\rho-p'k+\tfrac{1}{2}m;\,\sigma_1,\ldots,\sigma_r), \\ [(c):\psi',\ldots,\psi^{(r)}] : (1-\rho-p'k+\tfrac{1}{2}(m+r):\sigma_1,\ldots,\sigma_r), \end{matrix} \right.$$

$$\begin{matrix} (1-\rho-p'k-\tfrac{1}{2}m,\,\sigma_1,\ldots,\sigma_r) : (b',\phi');\ldots; \\ (-\rho-\tfrac{1}{2}(m+r)-p'k:\sigma_1,\ldots,\sigma_r):(d',\delta');\ldots; \end{matrix}$$

$$\left. \begin{matrix} (b^{(r)}:\phi^{(r)}) \\ ((d)^{(r)}:\delta^{(r)}) \end{matrix} ;\; z_1,\ldots,z_r \right). \qquad \ldots(2.4)$$

Now changing the order of integration and summation on the left-hand side of (2.4), which is justified (Carslaw 1930, p. 173), using the result (1.4) and, finally, replacing $\xi_j + \delta$ by ξ_j and $\eta_j + \delta$ by η_j, we obtain the value of the integral (2.2).

3. An Expansion Formula

In this section we derive the following expansion formula:

$$(1-x^2)^{\rho-1}\, {}_uF_v \left[\begin{matrix} \xi_u \\ \eta_v \end{matrix} ; h(1-x^2)^k \right] H\left[z_1(1-x^2)^{\sigma_1};\ldots;z_r(1-x^2)^{\sigma_r}\right]$$

$$= \sum_{n=0}^{\infty} J_2(\rho)\, S_{mn}(c,x) \qquad \ldots(3.1)$$

which is valid under the same conditions as those given in (2.2) with $\rho \geqslant 1$. $J_2(\rho)$ is the value of the integral defined by (2.2). The series on the right-hand side of (3.1) is assumed to be convergent.

Proof: From the general theory of Sturn-Liouville differential equations, it follows that the functions $S_{mn}(c, x)$ form the countably infinite orthonormal set complete in $(-1, 1)$. Hence any arbitrary function $f(x) \in C(-1, 1)$ can be represented as a linear combination of these functions, i.e.

$$f(x) = (1-x^2)^{\rho-1}\, {}_uF_v \left[\begin{matrix} \xi_u \\ \eta_v \end{matrix} , h(1-x^2)^k \right] H\left[z_1(1-x^2),\ldots,Z_r(1-x^2)^{\sigma_r}\right]$$

$$= \sum_{n=0}^{\infty} A_n S_{mn}(c,x),\; -1 < x < 1 \qquad \ldots(3.2)$$

where we have followed Churchill (1963, p. 57) and Taylor (1963, p. 111).

On multiplying both sides of (3.2) by $S_{mn}(c, x)$, integrating with respect to x over the interval $(-1, 1)$, and making use of the orthogonality property of spheroidal wave functions [Flammer 1957, p. 22, eqns. (3.1.32) and (3.1.33)], we get

$$J_2(\rho) = A_n \int_{-1}^{1} [S_{mn}(c, x)]^2 \, dx \quad \text{for } n' = n. \qquad \ldots(3.3)$$

Now, in order to avoid undesirable consequences in applications, we shall normalize the functions $S_{mn}(c, x)$ by the stipulation that

$$\int_{-1}^{1} [S_{mn}(c, x)]^2 \, dx = 1, \quad [(n - m) \text{ is even or odd}]$$

for all values of c.

Hence

$$A_n = J_2(\rho). \qquad \ldots(3.4)$$

Thus, by virtue of (3.2) and (3.4), the desired expansion formula (3.1) follows.

Remark : Regarding the convergence of the series on the right-hand sides of the results (2.1), (2.2) and (3.1), it would be worth mentioning that the ratio d_{r+2}^{mn}/d_r^{mn} is $-c^2/4r^2$ (Fox 1961, p. 17) and the ratio of gammas involving r and p is bounded for large values of r (even or odd) and p, by virtue of the fairly well-known result (*cf.* Erdélyi *et al.* 1953, p. 47)

$$\frac{\Gamma(r + \alpha)}{\Gamma(r + \beta)} = r^{\alpha-\beta} [1 + O(r^{-1})], \quad r \to \infty.$$

Hence the series on the right-hand side of (2.1), (2.2) and (3.1) are uniformly and absolutely convergent by the M-test.

4. Particular Cases

On specializing the parameters of the H-function of several variables of Srivastava and Panda (1976) in the results (2.1), (2.2) and (3.1), we can easily deduce various known results given earlier by Gupta and Sharma (1978), and Singh and Varma (1972).

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