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A NOTE ON SOME FUNDAMENTAL PARTIAL INTEGRAL INEQUALITIES

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The aim of the present note is to establish some fundamental partial integral inequalities which are n-independent variable generalizations of certain integral inequalities established by Pachpatte (1975). The inequalities established in this paper can be used in the analysis of a class of hyperbolic partial integral and integrodifferential equation as handy tools.

1. INTRODUCTION

One reason for much of the successful mathematical development in the theory of ordinary and partial differential equations is the availability of some kinds of inequalities and variational principles involving functions and their derivatives. Recently Pachpatte (1975) has proved some fundamental integral inequalities that have a wide range of applications in the theory of differential and integral equations of the more general type. Our objective here is to establish some integral inequalities which are the n-independent variable generalizations of the some integral inequalities established by Pachpatte (1975). The inequalities established here can be used to study the various problems in the theory of Hyperbolic partial integral and integrodifferential equations of the more general type.

2. MAIN RESULTS

Before giving the main results in this section we introduce the following notations (see Young 1973) which will be used in our subsequent discussion.

Let Ω be an open bounded set in R^n and let a point $(x_1, ..., x_n)$ in Ω be denoted by x. Let x^0 and $x(x^0 < x)$ be any two points in Ω, $\int_{x^0}^{x} \dots d\xi$ denotes the n-fold integral

$$\int_{x_1^0}^{x_1} \int_{x_2^0}^{x_2} \dots \int_{x_n^0}^{x_n} \dots d\xi_1 \dots d\xi_n, \; D_i = \frac{\partial}{\partial x_i}, \; 1 \leqslant i \leqslant n,$$

and denoted by D the parallelopiped defined by $x^0 < \xi < x$ (that is $x_i^0 < \xi_i < x_i$, $1 \leqslant i \leqslant n$). A useful generalization of Pachpatte's integral inequality (1975, Th. 1) in n independent variables is embodied in the following theorem.

Theorem 1 — Let $\phi(x)$, $p(x)$ and $q(x)$ be real-valued non-negative continuous functions defined on Ω, and $a(x)$ be a positive, monotonic, non-decreasing continuous function defined on Ω, for which the inequality

$$\phi(x) \leqslant a(x) + \int_{x^0}^{x} p(\xi)\, \phi(\xi)\, d\xi + \int_{x^0}^{x} p(\xi) \int_{x^0}^{\xi} q(\rho)\, \phi(\rho)\, d\rho)\, d\xi, \qquad \ldots(1)$$

holds for all $x \in \Omega$. Then

$$\phi(x) \leqslant a(x)\, [(1 + \int_{x^0}^{x} p(\xi)\,.\, \exp \{ \int_{x^0}^{\xi} (p(\rho) + q(\rho))\, d\rho\}\, d\xi]. \qquad \ldots(2)$$

PROOF : Since $a(x)$ is positive, monotonic, non-decreasing, we observe from (1) that

$$\frac{\phi(x)}{a(x)} \leqslant 1 + \int_{x^0}^{x} p(\xi)\, \frac{\phi(\xi)}{a(x)}\, d\xi + \int_{x^0}^{x} p(\xi) \left(\int_{x^0}^{\xi} q(\rho)\, \frac{\phi(\rho)}{a(x)}\, d\rho \right) d\xi$$

$$\leqslant 1 + \int_{x^0}^{x} p(\xi)\, \frac{\phi(\xi)}{a(\xi)}\, d\xi + \int_{x^0}^{x} p(\xi) \left(\int_{x^0}^{\xi} q(\rho)\, \frac{\phi(\rho)}{a(\rho)}\, d\rho \right) d\xi. \qquad \ldots(3)$$

Define

$$u(x) = 1 + \int_{x^0}^{x} p(\xi)\, \frac{\phi(\xi)}{a(\xi)}\, d\xi + \int_{x^0}^{x} p(\xi) \left(\int_{x^0}^{\xi} q(\rho)\, \frac{\phi(\rho)}{a(\rho)}\, d\rho \right) d\xi$$

$$u(x) = 1 \text{ on } x_j = x_j^0,\ 1 \leqslant j \leqslant n.$$

Then, differentiating $u(x)$, we have

$$D_1 \ldots D_n u(x) = p(x)\, \frac{\phi(x)}{a(x)} + p(x) \int_{x^0}^{x} q(\xi)\, \frac{\phi(\xi)}{a(\xi)}\, d\xi$$

which in view of (3) implies

$$D_1 \ldots D_n u(x) \leqslant p(x)\, (u(x) + \int_{x^0}^{x} q(\xi)\, u(\xi)\, d\xi). \qquad \ldots(4)$$

If we put

$$v(x) = u(x) + \int_{x^0}^{x} q(\xi) \,.\, u(\xi)\, d\xi \qquad ...(5)$$

$$v(x) = u(x) = 1 \text{ on } x_j = x_j^0,\ 1 \leqslant j \leqslant n.$$

Then

$$D_1 \ldots D_n v(x) = D, \ldots D_n u(x) + q(x)\, u(x) \qquad ...(6)$$

using (4) and the fact that $u(x) \leqslant v(x)$ from (5) in (6) we obtain

$$D_1 \ldots D_n v(x) \leqslant (p(x) + q(x))\, v(x)$$

i.e.

$$\frac{D_1 \ldots D_n v(x)}{v(x)} \leqslant p(x) + q(x). \qquad ...(7)$$

From (7) we observe that

$$\frac{v(x)\, D_1 \ldots D_n v(x)}{v^2(x)} \leqslant p(x) + q(x) + \frac{D_n v(x)\, D_1 \ldots D_{n-1} v(x)}{v^2(x)}$$

i.e.,

$$D_n \left(\frac{D_1 \ldots D_{n-1} v(x)}{v(x)} \right) \leqslant p(x) + q(x). \qquad ...(8)$$

By keeping $x_1 \ldots x_{n-1}$ fixed in the above inequality, set $x_n = \xi_n$ and then integrating both sides with respect to ξ_n from x_n^0 to x_n we have

$$\frac{D_1 \ldots D_{n-1} v(x)}{v(x)} \leqslant \int_{x_n^0}^{x_n} \{p(x_1 \ldots x_{n-1}, \xi_n) + q(x_1 \ldots x_{n-1}, \xi_n)\}\, d\xi_n. \qquad ...(9)$$

Again from (9) we observe that

$$\frac{v(x)\, D_1 \ldots D_{n-1} v(x)}{v^2(x)} \leqslant \int_{x_n^0}^{x_n} (p(x_1 \ldots x_{n-1}, \xi_n) + q(x_1 \ldots x_{n-1}, \xi_n))\, d\xi_n$$

$$+ \frac{D_{n-1} v(x) \,.\, D_1 \ldots D_{n-2} v(x)}{v^2(x)}$$

i.e.

$$D_{n-1} \left(\frac{D_1 \ldots D_{n-2} v(x)}{v^2(x)} \right) \leqslant \int_{x_n^0}^{x_n} (p(x_1 \ldots x_{n-1}\, \xi_n) + q(x_1 \ldots x_{n-1} \xi_n))\, d\xi_n.$$

...(10)

By keeping $x_1 \ldots x_{n-2}$, x_n fixed in the above inequality, set $x_{n-1} = \xi_{n-1}$ and then integrating with respect to ξ_{n-1} from x^0_{n-1} to x_{n-1}, we have

$$\frac{D_1 \ldots D_{n-2}v(x)}{v(x)} \leqslant \int_{x^0_{n-1}}^{x_{n-1}} \int_{x^0_n}^{x_n} (p(x_1 \ldots x_{n-2}\, \xi_{n-1}\xi_n) + q(x_1 \ldots x_{n-2}\xi_{n-1}\xi_n)) \times d\xi_{n-1} d\xi_n \qquad \ldots(11)$$

continuing in this way, we have

$$\frac{D_1 v(x)}{v(x)} \leqslant \int_{x^0_2}^{x_2} \ldots \int_{x^0_n}^{x_n} (p(x_1\xi_2 \ldots \xi_n) + q(x_1\xi_2 \ldots \xi_n))\, d\xi_2 \ldots d\xi_n. \qquad \ldots(12)$$

Now keeping $x_2 \ldots x_n$ fixed in (12), set $x_1 = \xi_1$ and then integrating with respect to ξ_1 from x^0_1 to x_1, we have

$$[\log v(x)]^{x_1}_{x^0_1} \leqslant \int_{x^0}^{x} (p(\xi) + q(\xi))\, d\xi$$

which implies

$$v(x) \leqslant \exp \int_{x^0}^{x} (p(\xi) + q(\xi))\, d\xi.$$

Substituting this in (4), we have

$$D_1 \ldots D_n u(x) \leqslant p(x) \exp \int_{x^0}^{x} (p(\xi) + q(\xi))\, d\xi. \qquad \ldots(13)$$

By iterated integration from x^0 to x, we obtain the bound on $u(x)$:

$$u(x) \leqslant 1 + \int_{x^0}^{x} p(\xi) \exp (\int_{x^0}^{\xi} (p(\rho) + q(\rho))\, d\rho)\, d\xi.$$

Substituting this bound on $u(x)$ in (3) we obtain the desired bound in (2). This completes the proof of the theorem.

We next establish the following n-independent variable generalization of the integral inequality established by Pachpatte (1975, Th. 2).

Theorem 2 — Let $\phi(x)$, $p(x)$, $q(x)$ and $h(x)$ be real-valued positive continuous functions defined on Ω; $W(u)$ be a positive, continuous monotonic, non-decreasing and submultiplicative function for $u > 0$, $W(0) = 0$ and suppose further that the inequality

$$\phi(x) \leqslant M + \int_{x^0}^{x} p(\xi)\, \phi(\xi)\, d\xi + \int_{x^0}^{x} p(\xi) \left(\int_{x^0}^{\xi} q(\rho)\, \phi(\rho)\, d\rho\right) d\xi$$

$$+ \int_{x^0}^{x} h(\xi) \, . \, W(\phi(\xi))\, d\xi \qquad \ldots(14)$$

is satisfied for all $x \in \Omega$, where M is a positive constant, then for $x^0 \leqslant x \leqslant x^*$,

$$\phi(x) \leqslant G^{-1} [G(M) + \int_{x^0}^{x} h(\xi)\, W(1 + \int_{x^0}^{\xi} p(\rho) \, . \, \exp (\int_{x^0}^{\rho} (p(\zeta) + q(\zeta)\, d\zeta)\, d\rho)\, d\xi]$$

$$\times [1 + \int_{x^0}^{x} p(\xi) \, . \, \exp (\int_{x^0}^{\xi} (p(\rho) + q(\rho))\, d\rho))\, d\xi],\ 0 \leqslant x \leqslant b \qquad \ldots(15)$$

where

$$G(r) = \int_{r^*}^{r} \frac{ds}{W(s)}\ r \geqslant r_0 > 0 \qquad \ldots(16)$$

and G^{-1} is the inverse function of G, and

$$G(M) + \int_{x^0}^{x} h(\xi)\, W(1 + \int_{x^0}^{\xi} p(\rho) \exp (\int_{x^0}^{\rho} (p(\zeta) + q(\zeta)\, d\zeta)\, d\rho)\, d\xi \in \text{Dom}\,(G^{-1}).$$

for all $x \in \Omega$ lying in the parallelepipod $x^0 \leqslant x \leqslant x^*$ in Ω.

PROOF : Define

$$a(x) = M + \int_{x^0}^{x} h(\xi)\, W(\phi(\xi))\, d\xi$$

$$a(0) = M.$$

Then (1) can be restated as

$$\phi(x) \leqslant a(x) + \int_{x^0}^{x} p(\xi)\, \phi(\xi)\, d\xi + \int_{x^0}^{x} p(\xi) \left(\int_{x^0}^{\xi} q(\rho)\, \phi(\rho)\, d\rho\right) d\xi$$

since $a(x)$ is positive, monotonic, non-decreasing on I, we have from Theorem 1

$$\phi(x) \leqslant a(x) [1 + \int_{x^0}^{x} p(\xi) \, . \, \exp (\int_{x^0}^{\xi} (p(\rho) + q(\rho))\, d\rho)\, d\xi]. \qquad \ldots(17)$$

Further

$$W(\phi(x)) \leqslant W(a(x)) \, . \, W(1 + \int_{x^0}^{x} p(\xi) \, . \, \exp (\int_{x^0}^{\xi} (p(\rho) + q(\rho))\, d\rho)\, d\xi).$$

Since W is submultiplicative. Hence

$$\frac{h(x)\; W(\phi(x))}{W(a(x))} \leqslant h(x)\; W\Big(1 + \int_{x^0}^{x} p(\xi) \exp\Big(\int_{x^0}^{\xi} (p(\rho) + q(\rho))\, d\rho\Big) d\xi\Big).$$

Because of (16) this reduces to

$$\frac{d}{dx}\, G(a(x)) \leqslant h(x)\; W\Big(1 + \int_{x^0}^{x} p(\xi) \exp\Big(\int_{x^0}^{\xi} (p(\rho) + q(\rho))\, d\rho\Big) d\xi\Big).$$

Now, integrating from 0 to x, we obtain

$$G(a(x)) - G(a(0)) \leqslant \int_{x^0}^{x} h(\xi)\; W(1 + \int_{x^0}^{\xi} p(\rho) \exp\, (\int_{x^0}^{\rho} (p(\zeta) + q(\zeta)\, d\zeta)\, d\rho)\, d\xi. \quad ...(18)$$

The desired bound in (15) follows from (17) and (18).

Remark : We note that there is no essential difficulty in establishing the n-independent variable generalizations of the other inequalities established by Pachpatte (1975) in view of the results established in Theorem 1 and 2. We omit the details.

3. An Application

In this section, we present a simple application of our Theorem 1 to study the boundedness of the solutions of a nonlinear hyperbolic partial integral equation in two independent variable of the form

$$u(x, y) = g(x, y) + \int_{x_0}^{x} \int_{y_0}^{y} F\,[x, y, s, t, u, \int_{x_0}^{s} \int_{y}^{t} k(s, t, \xi, \eta, u)\, d\xi\, d\eta]\, ds\, dt \quad ...(19)$$

where all the functions are continuous on their respective domains of their definitions and

$$|\, g(x, y)\, | \leqslant a(x, y) \quad ...(20)$$

$$|\, k(s, t, \xi, \eta, u)\, | \leqslant q(\xi, \eta)\, |\, u\, | \quad ...(21)$$

$$|\, F\,|\, x, y, s, t, u, \bar{u}\, \| \leqslant p(s, t)\, |\;\; |\, u\, | + |\, \bar{u}\, |\;\; | \quad ...(22)$$

for all $x \geqslant 0$, $y \geqslant 0$, where $a(x, y)$ is increasing in both the variables x and y and $p(x, y)$ and $q(x, y)$ are real-valued nonnegative continuous functions defined for $x \geqslant 0$, $y \geqslant 0$. Using (2) – (4) in (1) and a suitable application of Theorem 1 yields the bound on the solution $u(x, y)$ of (1) such that

$$| u(x, y) | \leqslant a(x, y) [1 + \int_{x_0}^{x} \int_{y_0}^{y} p(s, t) \times \exp (\int_{x_0}^{s} \int_{y_0}^{t} [p(\xi, \eta) + q(\xi, \eta)] \, d\xi \, d\eta) \, ds \, dt]. \quad ...(23)$$

Thus the right-hand side in (23) gives us the bound on the solution $u(x, y)$ of (1) in terms of the known functions.

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