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ON THE | B | AND | E, q | SUMMABILITY OF A FOURIER SERIES, ITS CONJUGATE SERIES AND THEIR DERIVED SERIES

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In this paper the Euler and Borel means are taken up to study the absolute summability of Fourier series and some associated series. Four theorems are established and it is also shown that these results are the best possible in a certain sense. The results of the paper and some classical results are used to deduce that like the ordinary summability methods (C, α) and (E, q), the absolute summability methods $| C, \alpha |$, $\alpha > 0$, and $| E, q |$, $q > 0$, are not comparable. It is also deduced from the theorems of the paper that the absolute Abel summability does not imply the absolute Borel summability.

§1. Let Σu_n be a given infinite series and $\{s_n\}$ be the sequence of its partial sums. Let

$$b(x) = e^{-x} \sum_{0}^{\infty} \frac{x^n}{n!} s_n.$$

Σu_n is said to be summable by Borel means to the sum s if $b(x) \to s$ as $x \to \infty$. It is said to be absolutely summable by Borel means and we write $\Sigma u_n \in | B |$, if

$$b(x) \in BV(0, \infty),$$

that is

$$\int_{0}^{\infty} e^{-x} \left| \sum_{0}^{\infty} \frac{x^n}{n!} u_{n+1} \right| dx < \infty.$$

Let

$$E_n(q) = \frac{1}{(q+1)^n} \sum_{0}^{n} \binom{n}{\nu} q^{n-\nu} s_\nu, \quad q \geqslant 0 \quad (E_n(0) = s_n).$$

If $E_n(q) \to s$ as $n \to \infty$, Σu_n is said to be summable to s by the Euler method (E, q). The series is said to be absolutely summable by the method (E, q), and we write $\Sigma u_n \in |E, q|$, $q \geqslant 0$, if $\{E_n(q)\} \in bv$.

It is known that when $-1 < q \leqslant q'$, $|E, q| \subseteq |E, q'| \subseteq |B|$ (see Knopp and Lorentz 1949–50).

Let $f \in L(-\pi, \pi)$ and be a periodic function with period 2π. Let the Fourier series of f be given by

$$f(x) \sim \tfrac{1}{2} a_0 + \sum_{1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{0}^{\infty} A_n(x). \qquad \text{...(1.1)}$$

The series conjugate to (1.1) is

$$\sum_{1}^{\infty} (b_n \cos nx - a_n \sin nx) = \sum_{1}^{\infty} B_n(x). \qquad \text{...(1.2)}$$

The derived series of (1.1) and (1.2) are respectively

$$\Sigma\, nB_n(x) \qquad \text{...(1.3)}$$

and

$$-\Sigma\, nA_n(x). \qquad \text{...(1.4)}$$

We write

$$\phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t) - 2s\}$$

$$\psi(t) = \tfrac{1}{2}\{f(x+t) - f(x-t)\}$$

$$\phi_1(t) = \frac{1}{t}\int_0^t \phi(u)\, du$$

and

$$\psi_1(t) = \frac{1}{t}\int_0^t \psi(u)\, du.$$

K, K_1, K_2 denote absolute constants.

The purpose of this paper is to study the absolute summability by the Euler and Borel means of the Fourier series (1.1), its conjugate series (1.2), and their derived series (1.3) and (1.4), and to establish some general theorems. We show that some known results on summability $|E, q|$ of Fourier series and its conjugate series may not be improved upon in a certain sense. Actually, the following results are known:

Theorem A — Let $0 < \delta < 1$ and $q > 0$. Then

(i) $\phi(t) \log 1/t \in BV(0, \delta) \Rightarrow \Sigma A_n(x) \in | E, q |$;

(ii) $\psi(t) \log 1/t \in BV(0, \delta)$ and $\psi(t)/t \in L(0, \delta) \Rightarrow \Sigma B_n(x) \in | E, q |$;

(iii) $\phi_1(t)/t \in BV(0, \delta) \Rightarrow \Sigma A_n(x) \in | E, q |$; and

(iv) $\psi_1(t)/t \in BV(0, \delta) \Rightarrow \Sigma B_n(x) \in | E, q |$.

The results in (i) and (ii) are given by Mohanty and Mohapatra (1968) and independently by Kwee (1972). The results in (iii) and (iv) are special cases of more general results given elsewhere (Chandra 1977).

In Theorems 1 and 2 we show that even a slight relaxation in the conditions on the generating function in (i) and (ii) of Theorem A, may result in the non-summability of the respective series even by the method $| B |$.

It can be easily seen that for $0 < \delta < \pi$,

$$\phi(t)/t^2 \in L(0, \delta) \Rightarrow \phi_1(t)/t \in BV(0, \delta)$$

and that the converse is not necessarily true. Theorem 3 shows that results (iii) and (iv) of Theorem A are the best possible in the sense that even a slight relaxation on stronger assumptions than those in these results, may result in non-summability even by the method $| B |$.

In Theorem 4 we provide sufficient conditions for summability $| E, q |$ of the first derived series of a Fourier series and its conjugate series. Again by producing results similar to Theorems 1–3, we show that even here the hypotheses on the generating function may not be relaxed too much.

Some of the results proved here improve upon the corresponding results due to Tripathy (1969) and Kwee (1972). A result of Whittaker (1930) gives us that $\phi(t)/t \in L(0, \pi) \Rightarrow \Sigma A(x) \in | A |$, that is, it is absolutely summable by Abel's method. After Theorem 3 we then deduce that summability $| A |$ does not necessarily imply summability $| B |$.

Indeed, we establish the following theorems:

Let function p, defined over $(0, \pi)$, be such that

$$p(t) \neq 0, \text{ for } t \in (0, \pi) \text{ and } p(t) = o(1), \text{ as } t \to 0. \qquad \ldots(1.5)$$

Theorem 1 — If p satisfies (1.5) then $\phi(t) \in AC(0, \pi)$, and

$$\int_0^\pi \log \frac{2\pi}{t} \, | p(t) \, \phi'(t) | \, dt < \infty$$

do not ensure summability $| B |$ of (1.1).

Theorem 2 — If p satisfies (1.5) then $\psi(t) \in AC(0, \pi)$, $\psi(0+) = 0$ and

$$\int_0^\pi \log \frac{2\pi}{t} \mid p(t)\psi'(t) \mid dt < \infty$$

do not ensure the summability $| B |$ of the conjugate series (1.2).

Theorem 3 — Suppose p satisfies (1.5). Then

(i) $p(t)\, t^{-2}\phi(t) \in L(0, \pi)$ is not sufficient to ensure the summability $| B |$ of the series (1.1); and

(ii) $p(t)\, t^{-2}\psi(t) \in L(0, \pi)$ is not sufficient to ensure the summability $| B |$ of the series (1.2).

Theorem 4 — Let $0 < \delta < \pi$ and $q > 0$. Then

(i) $t^{-2}\psi(t) \in BV(0, \delta) \Rightarrow \Sigma\, nB_n(x) \in | E, q |$, and

(ii) $t^{-2}\phi(t) \in BV(0, \delta) \Rightarrow \Sigma\, nA_n(x) \in | E, q |$, $q > 0$.

However, if p satisfies (1.5) then

(iii) even $\psi(t) \in AC(0, \pi)$ and $\int_0^\pi t^{-2} \mid p(t)\psi'(t) \mid dt < \infty$ do not ensure even the summability $| B |$ of the derived series $\Sigma\, nB_n(x)$; and

(iv) even $\phi(t) \in AC(0, \pi)$ and $\int_0^\pi t^{-2} \mid p(t)\phi'(t) \mid dt < \infty$ do not ensure even the summability $| B |$ of the derived series $\Sigma\, nA_n(x)$.

§2. We shall make use of the following results for the proof of our theorems.

Lemma 1 — (Sunouchi and Tsuchikura 1952). Let F be measurable over $(0, \infty) \times (0, \infty)$. In order that for every $g \in L(0, \pi)$, the function

$$G(x) = \int_0^\infty F(x, t)\, g(t)\, dt$$

should be defined almost everywhere and

$$\int_0^\infty \mid G(x) \mid dx < \infty$$

it is necessary and sufficient that

$$\operatorname*{ess\ sup}_{0<t<\infty} \int_0^\infty | F(x, t) | \, dx < \infty.$$

Lemma 2 — Let $q > 0$. Then for $0 < t \leqslant \pi$,

(i) $$\left(\log \frac{k\pi}{t}\right)^{-1} \sum_0^\infty (q+1)^{-n-1} \left| \sum_0^n \binom{n}{k} q^{n-k} \frac{e^{ikt}}{(k+1)} \right| = O(1);$$

(ii) $$t^2 \sum_0^\infty (q+1)^{-n-1} \left| \sum_0^n \binom{n}{k} q^{n-k} e^{ikt} \right| = O(1),$$

(iii) $$t^4 \sum_0^\infty (q+1)^{-n-1} \left| \sum_0^n \binom{n}{k} q^{n-k} (k+1) e^{i(k+1)t} \right| = O(1).$$

PROOF : Let $\rho = (1 + q^2 + 2q \cos t)^{1/2}$. Then

(i) $$\sum_0^\infty (q+1)^{-n-1} \left| \sum_0^n \binom{n}{k} q^{n-k} \frac{e^{ikt}}{k+1} \right|$$

$$\leqslant \sum_0^\infty \frac{1}{(n+1)} \{\rho/(1+q)\}^{n+1} + \sum_0^\infty \frac{1}{(n+1)} \left(\frac{q}{1+q}\right)^{n+1}$$

$$= \log \{(1+q)/(1+q-\rho)\} + \log (1+q)$$

$$= O\left(\log \frac{2\pi}{t}\right);$$

(ii) $$\sum_0^\infty (q+1)^{-n-1} \left| \sum_0^n \binom{n}{k} q^{n-k} e^{ikt} \right|$$

$$= \frac{1}{(q+1)} \sum_0^\infty \left(\frac{\rho}{1+q}\right)^n,$$

$$= O\left(\frac{1}{t^2}\right);$$

and

(iii) $$\sum_0^\infty (q+1)^{-n-1} \left| \sum_0^n \binom{n}{k} q^{n-k} (k+1) e^{i(k+1)t} \right|$$

(*equation continued on p.* 1355)

$$\leqslant \frac{1}{(q+1)} \sum_{0}^{\infty} \left(\frac{\rho}{1+q}\right)^{n} + \frac{1}{(q+1)^{2}} \sum_{1}^{\infty} n \left(\frac{\rho}{1+q}\right)^{n-1},$$

$$= O(1/t^{4}).$$

§3. *Proof of Theorem* 1 — We have

$$A_n(x) = \frac{2}{\pi} \int_0^{\pi} \phi(t) \cos nt \, dt$$

$$= -\frac{2}{n\pi} \int_0^{\pi} \sin nt \, \phi'(t) \, dt.$$

Let $$g_1(t) = p(t) \log \frac{2\pi}{t} \phi'(t).$$

Then $$\sum A_n(x) \in | B |, \text{ if and only if}$$

$$\int_0^{\infty} e^{-y} \left| \sum_0^{\infty} \frac{y^n}{n!} A_{n+1}(x) \right| dy$$

$$= \frac{2}{\pi} \int_0^{\infty} e^{-y} \left| \sum_0^{\infty} \frac{y^n}{n!} \int_0^{\pi} \frac{\sin (n+1)t}{(n+1)} \phi'(t) \, dt \right| dy$$

$$= \frac{2}{\pi} \int_0^{\infty} e^{-y} \left| \int_0^{\pi} \sum_0^{\infty} \frac{y^n}{(n+1)!} \sin (n+1)t \, \phi'(t) \, dt \right| dy$$

$$= \frac{2}{\pi} \int_0^{\infty} e^{-y} \left| \int_0^{\pi} \frac{1}{p(t) \log \frac{2\pi}{t}} \sum_0^{\infty} \frac{y^n}{(n+1)!} \sin (n+1)t \, g_1(t) \, dt \right| dy$$

$$< \infty.$$

By Lemma 1, this holds for all $g_1 \in L(0, \pi)$ if and only if

$$\operatorname*{ess\,sup}_{0<t<\pi} \frac{1}{| p(t) | \log \frac{2\pi}{t}} \int_0^{\infty} e^{-y} \left| \sum_0^{\infty} \frac{y^n}{(n+1)!} \sin (n+1)t \right| dy < \infty.$$

However,

$$\frac{1}{|p(t)| \log \frac{2\pi}{t}} \int_0^\infty e^{-y} \left| \sum_0^\infty \frac{y^n}{(n+1)!} \sin (n+1)t \right| dy$$

$$= \frac{1}{|p(t)| \log \frac{2\pi}{t}} \int_0^\infty \frac{e^{-y}}{y} \mid e^{y \cos t} \sin (y \sin t) \mid dy$$

$$= \frac{1}{|p(t)| \log \frac{2\pi}{t}} \int_0^\infty \frac{e^{-u \tan t/2}}{u} \mid \sin u \mid du$$

$$= \frac{1}{|p(t)| \log \frac{2\pi}{t}} \sum_1^\infty \int_{(n-1)\pi}^{n\pi} \frac{e^{-u \tan t/2}}{u} \mid \sin u \mid du$$

$$\geqslant \frac{1}{\pi |p(t)| \log \frac{2\pi}{t}} \sum_1^\infty \frac{e^{-n\pi \tan t/2}}{n} \left| \int_{(n-1)\pi}^{n\pi} \sin u \, du \right|$$

$$= \frac{2}{\pi |p(t)| \log \frac{2\pi}{t}} \sum_1^\infty \frac{e^{-(\pi \tan t/2)n}}{n}$$

$$= \frac{2}{\pi |p(t)| \log \frac{2\pi}{t}} \log (1 - e^{-\pi \tan t/2})^{-1}$$

$$\to +\infty, \text{ as } t \to 0,$$

and this completes the proof of the theorem.

Proof of Theorem 2 — Note that

$$B_n(x) = \frac{2}{\pi} \int_0^\pi \psi(t) \sin nt \, dt = \frac{2}{n\pi} \int_0^\pi \cos nt \, \psi'(t) \, dt.$$

We write $g_2(t) = p(t) \log \frac{2\pi}{t} \psi'(t)$. Arguing as in Theorem 1, we see that, in order that $\Sigma B_n(x) \in |B|$ for every $g_2 \in L(0, \pi)$ it is necessary that

$$\operatorname*{ess\,sup}_{0<t<\pi} \frac{1}{|p(t)| \log \frac{2\pi}{t}} \int_0^\infty e^{-y} \left| \sum_0^\infty \frac{y^n}{(n+1)!} \cos (n+1)t \right| dy < \infty.$$

However, as in the proof of Theorem 1, we have

$$\int_0^\infty e^{-y} \left| \sum_0^\infty \frac{y^n}{(n+1)!} \cos (n+1)t \right| dy$$

$$= \int_0^\infty \left| \frac{e^{-y(1-\cos t)} \cos (y \sin t)}{y} - \frac{e^{-y}}{y} \right| dy$$

$$\geqslant \sum_1^\infty \int_{(n-\frac{1}{2})\pi \operatorname{cosec} t}^{(n+\frac{1}{2})\pi \operatorname{cosec} t} \frac{1}{y} \left| e^{-y \sin t \tan (t/2)} \cos (y \sin t) - e^{-y} \right| dy$$

$$\geqslant \sum_1^\infty \left| \int_{(n-\frac{1}{2})\pi \operatorname{cosec} t}^{(n+\frac{1}{2})\pi \operatorname{cosec} t} y^{-1} e^{-y \sin t \tan (t/2)} \cos (y \sin t)\, dy \right|$$

$$- \int_{\frac{1}{2}\pi \operatorname{cosec} t}^{\infty} \frac{e^{-y}}{y}\, dy$$

$$\geqslant \sum_1^\infty \frac{e^{-(n+(1/2))\pi \tan t/2}}{(n+\frac{1}{2})\pi} \left| \int_{(n-\frac{1}{2})\pi}^{(n+\frac{1}{2})\pi} \cos u\, du \right| - O(1)$$

$$\geqslant \frac{2}{\pi} \log (1 - e^{-\pi \tan t/2})^{-1} - \frac{2}{\pi} e^{-\pi \tan t/2} - O(1).$$

$$\Rightarrow \operatorname*{ess\,sup}_{0<t<\pi} \left(\frac{1}{|p(t)| \log \frac{2\pi}{t}} \int_0^\infty e^{-y} \left| \sum_0^\infty \frac{y^n \cos (n+1)t}{(n+1)!} \right| dy \right) = \infty,$$

and this completes the proof of the theorem.

Proof of Theorem 3 — Write $g_3(t) = t^{-2} p(t)\, \phi(t)$. Then $\Sigma A_n(x) \in |B|$, if and only if

$$\int_0^\infty e^{-y} \left| \sum_0^\infty \frac{y^n}{n!} \int_0^\pi \cos (n+1)t\, \phi(t)\, dt \right| dy$$

$$= \int_0^\infty e^{-y} \left| \int_0^\pi \sum_0^\infty \frac{y^n}{n!} \cos (n+1)t\, \phi(t)\, dt \right| dy$$

(*equation continued on p.* 1358)

$$= \int_0^\infty e^{-y} \left| \int_0^\pi \frac{t^2}{p(t)} \sum_0^\infty \frac{y^n}{n!} \cos(n+1)\, t\, g_3(t)\, dt \right| dy$$

$$< \infty.$$

By Lemma 1, this holds for all $g_3 \in L(0, \pi)$ if and only if

$$\operatorname*{ess\,sup}_{0<t<\pi} \left(\frac{t^2}{|\,p(t)\,|} \int_0^\infty e^{-y} \left| \sum_0^\infty \frac{y^n}{n!} \cos(n+1)t \right| dy \right) < \infty.$$

Now

$$\int_0^\infty e^{-y} \left| \sum_0^\infty \frac{y^n}{n!} \cos(n+1)t \right| dy = \int_0^\infty e^{y(\cos t - 1)} \,|\cos(y \sin t + t)|\, dy$$

$$= \frac{1}{\sin t} \int_t^\infty e^{-(u-t)\tan(t/2)} \,|\cos u|\, du$$

$$> \operatorname{cosec} t \sum_2^\infty \int_{(n-\frac{1}{2})\pi}^{(n+\frac{1}{2})\pi} e^{-(u-t)\tan(t/2)} \,|\cos u|\, du$$

$$\geqslant \operatorname{cosec} t \sum_2^\infty e^{-\{(n+(1/2))\pi - t\}\tan(t/2)} \left| \int_{(n-\frac{1}{2})\pi}^{(n+\frac{1}{2})\pi} \cos u \, du \right|$$

$$\geqslant 2 \operatorname{cosec} t\, e^{(t-(\pi/2))\tan(t/2)} \sum_2^\infty e^{-n\pi \tan(t/2)}$$

$$= 2 \operatorname{cosec} t\, e^{(t-(5\pi/2))\tan(t/2)} (1 - e^{-\pi \tan(t/2)})^{-1}.$$

Therefore

$$\operatorname*{ess\,sup}_{0<t<\pi} \left(\frac{t^2}{|\,p(t)\,|} \int_0^\infty e^{-y} \left| \sum_0^\infty \frac{y^n}{n!} \cos(n+1)t \right| dy \right) = \infty. \qquad \text{...(3.1)}$$

This proves the first part.

Similarly, we note that $\Sigma B_n(x) \in |B|$, if and only if

$$\operatorname*{ess\,sup}_{0<t<\pi} \left(\frac{t^2}{|\,p(t)\,|} \int_0^\infty e^{-y} \left| \sum_0^\infty \frac{y^n}{n!} \sin(n+1)t \right| dy \right) < \infty.$$

Now proceeding as in the previous case and Theorem 1, we get the desired result and the proof of the theorem is completed.

Proof of Theorem 4 — It follows at once from Lemma 2(iii) that the summability $|E, q|$ of the differentiated Fourier series, or the differentiated conjugate series, is a local property so that there is no loss of generality in taking $\psi(t) = 0$ (or $\phi(t) = 0$) in (δ, π). We then have

$$nB_n(x) = \frac{2n}{\pi} \int_0^\delta \psi(t) \sin nt \, dt$$

$$= \frac{2n}{\pi} \left\{ \frac{\psi(\delta)}{\delta^2} \right\} \int_0^\delta u^2 \sin nu \, du - \int_0^\delta \int_0^t u^2 \sin nu \, du \, d(\psi(t)/t^2) \Bigg\}.$$

There is a similar result for $nA_n(x)$ with sines replaced by cosines and ψ replaced by ϕ. Thus both results will follow if we prove that, uniformly in $0 < t \leqslant \delta$,

$$\sum_0^\infty (q+1)^{-n-1} \left| \sum_0^n \binom{n}{k} q^{n-k}(k+1) \int_0^t u^2 \exp\{i(k+1)u\} \, du \right| = O(1).$$

However, this follows after necessary integrations by parts, from Lemma 2 and the regularity of the method $|E, q|$, $q \geqslant 0$. This proves the first part of the theorem.

Write $g_4(t) = t^{-2} p(t)\psi'(t)$ and $g_5(t) = t^{-2} p(t)\phi'(t)$. As

$$nB_n(x) = \frac{2}{\pi} \int_0^\pi \cos nt \, \psi'(t) \, dt,$$

we get that $\Sigma \, nB_n(x) \in |B|$, if and only if

$$\int_0^\infty e^{-y} \left| \sum_0^\infty \frac{y^n}{n!} \int_0^\pi \cos (n+1)t \, \psi'(t) \, dt \right| dy$$

$$= \int_0^\infty e^{-y} \left| \int_0^\pi \frac{t^2}{p(t)} \sum_0^\infty \frac{y^n}{n!} \cos (n+1)t \, g_4(t) \, dt \right| dy < \infty.$$

Similarly, $\Sigma \, nA_n(x) \in |B|$, if and only if

$$\int_0^\infty e^{-y} \left| \int_0^\pi \frac{t^2}{p(t)} \sum_0^\infty \frac{y^n}{n!} \sin (n+1)t \, g_5(t) \, dt \right| dy < \infty.$$

However, this is what has already been discussed in the proof of Theorem 3, and thus the theorem is completely proved.

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