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TWO DIMENSIONAL STEADY STATE TEMPERATURE DISTRIBUTION IN SKIN AND SUBCUTANEOUS TISSUES

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Temperature distribution has been studied in skin and subcutaneous part of human body exposed to a cool and dry environment with negligible insensible perspiration. The mathematical model incorporates the effect of blood mass flow and metabolic heat generation. The blood mass flow rate, metabolic heat generation and tissue thermal conductivity are assumed to be have different values in all the three parts of the region. Variational finite element method has been used to obtain numerical values of the nodal temperatures for a wide range of skin surface temperature and for low atmospheric temperature. Matrix notations have been used to represent the problem. Firstly, the mathematical formulation has been done for a general element and then with the help of this formulation the matrix for the whole region has been obtained. It has been observed that temperature of outer layer of skin is influenced by skin surface temperature and temperatures of inner layers are mainly effected by body core temperature.

INTRODUCTION

The skin and subcutaneous region play an important role in the temperature control of the human body. The skin consists of two layers; the epidermis and dermis. The epidermis is composed of two layers of cells. The outer of these layers, stratum corneum, is made up all of non-living material keratin. This layer is hard and flattened. The lower layer, stratum germinativum, is division of cells.

The dermis is composed of matted masses of connective tissue and elastic fibres through which pass numerous blood vessels, lymphatics and nerves. The sub-dermal tissues contain fat cells, connective tissue, blood vessels, lymphatics and nerves. Smooth and striated muscle fibres are also found in or passing through subcutaneous connective tissue. This region serves primarily as connection between the skin and the deeper tissues.

The skin layers have different physical properties. Also these layers are neither smooth nor distinct from each other. Sometimes the geometry of each layer is very irregular. The lower portion of stratum germinativum is very irregular in outline due to many upward projections of the dermis.

Due to complexity of structure and involvement of many physical parameters, a theoretical study of this problem may be quite useful. Based on earlier experimental results Perl (1962) introduced a mathematical formulation of heat transfer problems in living biological system. Attempts have been made by Patterson (1976) for experimental determination of temperature profiles in SST region. Recently, Saxena (1978) and Saxena and Arya (1979, 1981) have obtained temperature distribution in SST for one dimensional case by using analytic and numerical methods. All the above studies have been made for one dimensional cases only.

In the present study a rectangular area of the skin has been considered to be exposed to cool and dry environment. As usual skin is considered to have three layers, namely, epidermis, dermis and subcutaneous tissues (SST). Each layer has been divided into eight triangular elements. In this way the whole region consists of twenty four triangular elements and twenty one nodal points. Average values of blood mass flow rate, metabolic heat generation and thermal conductivities have been considered for each layer.

Theoretical Formulation

If θ denotes temperature at a distance x from the outer skin surface and at time t then in terms of partial derivatives the three components of rate of temperature change with respect to time, expressing perfusion (p), diffusion (d) and metabolism (m) can be summed up in the following form:

$$\frac{\partial \theta}{\partial t} = \left.\frac{\partial \theta}{\partial t}\right|_p + \left.\frac{\partial \theta}{\partial t}\right|_d + \left.\frac{\partial \theta}{\partial t}\right|_m . \qquad \ldots(1)$$

The first two terms on the right-hand side can be elaborated by using Fick's Perfusion Principle and Fourier's Law for diffusion respectively and the third term may be represented by a quantity S denoting rate of heat generation per unit volume (Perl 1962). Consequently, the following differential equation is obtained:

$$\rho c \frac{\partial \theta}{\partial t} = \text{div} \, (K \, \text{grad} \, \theta) + m c_b \, (\theta_b - \theta) + S \qquad \ldots(2)$$

where ρ, c and K are density, specific heat and thermal conductivity of tissue respectively. m and c_b are mass flow rate and specific heat of the blood respectively. θ_b is the temperature of the blood when it enters the tissue element.

The partial differential equation (2), when written in two dimensional steady state case and compared with the Euler-Lagrange equation (Myers 1971), gives the following variational statement for the problem:

$$I = \tfrac{1}{2} \int_{y=0}^{y_s} \int_{x=0}^{a} \left[K \left(\frac{\partial \theta}{\partial x} \right)^2 + K \left(\frac{\partial \theta}{\partial y} \right)^2 + m c_b (\theta_b - \theta)^2 - 2S\theta \right] dx \, dy. \qquad \ldots(3)$$

Integral I may be divided into three parts by writting equation (3) as:

$$I = I_k + I_h - I_g \qquad \ldots(4)$$

where
$$I_k = \tfrac{1}{2} \iint_A \left[K \left(\frac{\partial \theta}{\partial x}\right)^2 + K \left(\frac{\partial \theta}{\partial y}\right)^2 \right] dx\, dy \qquad \ldots(5)$$

$$I_h = \tfrac{1}{2} \iint_A mc_b(\theta_b - \theta)^2\, dx\, dy, \quad I_g = \iint_A S\theta\, dx\, dy \qquad \ldots(6)$$

and A denotes the region of interest.

Now, I is to be minimized by differentiating it with respect to each nodal point temperature and setting the derivatives equal to zero. That is:

$$\frac{dI}{d\Theta} = 0 = \frac{dI_k}{d\Theta} + \frac{dI_h}{d\Theta} - \frac{dI_g}{d\Theta}. \qquad \ldots(7)$$

The region contains twenty four elements so that the differentials in eqn. (7). are computed as a sum over each of the small elements. Thus

$$\frac{dI_k}{d\Theta} = \sum_{e=1}^{24} \frac{dI_k^{(e)}}{d\Theta}, \quad \frac{dI_h}{d\Theta} = \sum_{e=1}^{24} \frac{dI_h^{(e)}}{d\Theta}, \quad \frac{dI_g}{d\Theta} = \sum_{e=1}^{24} \frac{dI_g^{(e)}}{d\Theta} \qquad \ldots(8)$$

here (e) denotes a particular element. First of all we obtain the value of $\frac{dI_k^{(e)}}{d\Theta}$. For a particular element (e) (Fig. 1), $I_k^{(e)}$ will be function of three corner temperatures θ_i θ_j and θ_k only. Thus the partial derivative of $I_k^{(e)}$ with respect to θ is a column matrix of mostly zero elements because $I_k^{(e)}$ depends only on θ_i, θ_j and θ_k and not on the other nodal temperatures. That is

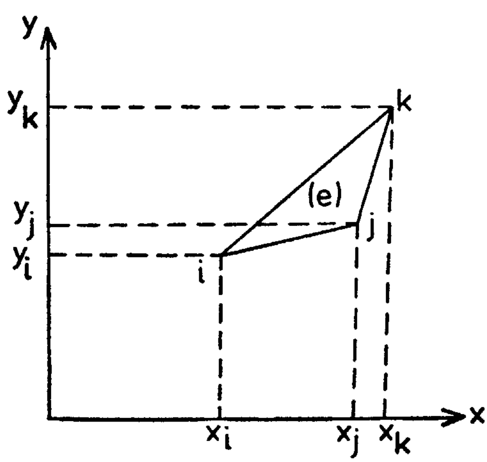

FIG. 1. Finite element triangular element.

$$\frac{dI_k^{(e)}}{d\Theta} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \dfrac{\partial I_k^{(e)}}{\partial \theta_i} \\ \dfrac{\partial I_k^{(e)}}{\partial \theta_j} \\ \dfrac{\partial I_k^{(e)}}{\partial \theta_k} \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ i\text{th row} \\ j\text{th row} \\ k\text{th row} \\ \\ \\ \\ \\ \end{matrix} \qquad \ldots(9)$$

To avoid working with all these zeros, we consider this 21×1 column matrix to be the product of an 21×3 matrix and a 3×1 column matrix. That is, we can write now

$$\frac{dI_k^{(e)}}{d\Theta} = M^{(e)} \frac{dI_k^{(e)}}{d\Theta^{(e)}} \qquad \ldots(10)$$

where

$$M^{(e)} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ i\text{th row} \\ j\text{th row} \\ k\text{th row} \\ \\ \end{matrix} \qquad \ldots(11)$$

$$\frac{dI_k^{(e)}}{d\Theta^{(e)}} = \begin{bmatrix} \dfrac{\partial I_k^{(e)}}{\partial \theta_i} \\ \dfrac{\partial I^{(e)}}{\partial \theta_j} \\ \dfrac{\partial I^{(e)}}{\partial \theta_k} \end{bmatrix}. \qquad \ldots(12)$$

Similarly matrices for $\dfrac{dI_h^{(e)}}{d\Theta}$ and $\dfrac{dI_g^{(e)}}{d\Theta}$ may be reduced to the form similar to (10).

Linear variation (Myers 1971) of temperature within each element is expressed as follows:

$$\theta^{(e)} = C_1^{(e)} + C_2^{(e)} x + C_3^{(e)} y \qquad \text{...(13)}$$

where $\theta^{(e)}$ is equal to θ_i, θ_j and θ_k at the corners of the element. Equation (13) may be written in matrix form as

$$\Theta^{(e)} = P^{(e)} C^{(e)} \qquad \text{...(14)}$$

where

$$\Theta^{(e)} = \begin{bmatrix} \theta_i \\ \theta_j \\ \theta_k \end{bmatrix}, \qquad C^{(e)} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}^{(e)}$$

$$P^{(e)} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix}.$$

Equation (13) may also be written as

$$\theta^{(e)} = p^T C^{(e)} = p^T R^{(e)} \Theta^{(e)} \qquad \text{...(15)}$$

where

$$R^{(e)} = P^{(e)-1} \text{ and } p^T = [1 \; x \; y].$$

Matrix $R^{(e)}$ has the following value:

$$R^{(e)} = \frac{1}{x_{ij} y_{jk} - x_{jk} y_{ij}} \left[\begin{array}{c|c|c} x_j y_k - x_k y_j & x_k y_i - x_i y_k & x_i y_j - x_j y_i \\ -y_{jk} & y_{ik} & -y_{ij} \\ x_{jk} & -x_{ik} & x_{ij} \end{array} \right]$$

here $x_{ij} = x_j - x_i$ and $y_{ij} = y_j - y_i$. Now eqn. (5) may be written as:

$$I^{(e)} = \frac{K^{(e)}}{2} \iint_{A^{(e)}} [(p_x^T R^{(e)} \Theta^{(e)})^2 + (p_y^T R^{(e)} \Theta^{(e)})^2] \, dx \, dy. \qquad \text{...(16)}$$

Differentiating (16) with respect to each of the three corner temperatures at the same time, we get

$$\frac{dI_k^{(e)}}{d\Theta^{(e)}} = K^{(e)} R^{(e)T} \iint_{A^{(e)}} (p_x p_x^T + p_y p_y^T) \, dx \, dy \, R^{(e)} \Theta^{(e)}. \qquad \text{...(17)}$$

Detailed calculation of the right-hand side gives

$$\frac{dI_k^{(e)}}{d\Theta^{(e)}} = B^{(e)}\Theta^{(e)} \qquad \ldots(18)$$

where

$$B^{(e)} = \frac{K^{(e)}A^{(e)}}{(x_{ij}y_{jk} - x_{jk}y_{ij})^2} \times \begin{bmatrix} x_{jk}^2 + y_{jk}^2 & -(x_{ik}x_{jk} + y_{ik}y_{jk}) & x_{ij}x_{jk} + y_{ij}y_{jk} \\ & x_{ik}^2 + y_{ik}^2 & -(x_{ij}x_{ik} + y_{ij}y_{ik}) \\ (\text{symmetric}) & & x_{ij}^2 + y_{ij}^2 \end{bmatrix}$$

$$A^{(e)} = \tfrac{1}{2}\,|\,x_{ij}y_{jk} - x_{jk}y_{ij}\,|\,.$$

Now, from eqns. (8) and (10) we get

$$\frac{dI_k}{d\Theta} = \sum_{e=1}^{24} M^{(e)}B^{(e)}\Theta^{(e)} = \sum_{e=1}^{24} M^{(e)}B^{(e)}M^{(e)T}\,\Theta = B\Theta \qquad \ldots(19)$$

where

$$\Theta^{(e)} = M^{(e)T}\,\Theta, \quad B = \sum_{e=1}^{24} M^{(e)}B^{(e)}M^{(e)T}.$$

Similarly we obtain

$$\frac{dI_g}{d\Theta} = g\sum_{e=1}^{24} S^{(e)} \qquad \ldots(20)$$

$$\frac{dI_h}{d\Theta} = (H\Theta - g\theta_b)\sum_{e=1}^{24} (mc_b)^{(e)} \qquad \ldots(21)$$

where

$$g = \sum_{e=1}^{24} M^{(e)}g^{(e)},\; g^{(e)} = \frac{A^{(e)}}{3}[1 \quad 1 \quad 1]^T$$

$$H = \sum_{e=1}^{24} M^{(e)}H^{(e)},\; H^{(e)} = R^{(e)T}A^{(e)}Q_c^{(e)}\,R^{(e)}$$

$$Q_c^{(e)} = \begin{bmatrix} 1 & x_c & y_c \\ x_c & A^{(e)}x_c^2 & A^{(e)}x_cy_c \\ y_c & A^{(e)}x_cy_c & A^{(e)}y_c^2 \end{bmatrix}^{(e)}$$

$$x_c = \tfrac{1}{3}(x_i + x_j + x_k), \quad y_c = \tfrac{1}{3}(y_i + y_j + y_k).$$

Substitution of eqns. (19), (20) and (21) into eqn. (7) gives

$$\frac{dI}{d\Theta} = B\Theta + (H\Theta - g\theta_b) \sum_{e=1}^{24} (mc_b)^{(e)} - g \sum_{e=1}^{24} S^{(e)} = 0$$

or

$$[B + H \sum_{e=1}^{24} (mc_b)^{(e)}]\, \Theta = [\sum_{e=1}^{24} (mc_b\theta_b + S)^{(e)}]\, g. \qquad \ldots(22)$$

Equation (22), if written in matrix form, gives a matrix of order 21×21. Keeping inview that outer skin surface temperature is prescribed and body core temperature is constant (37°C), this matrix reduces to order 15×15. Solution of this final matrix gives the values of remaining nodal temperatures.

Numerical Results

From the geometry of the skin it appears that there may be some blood flow in the cells which are situated at the interface of epidermis and dermis. Some part of the dermis is projected inside the epidermis layer and this part will have some blood vessels. Population density of the blood vessels increases gradually and becomes uniform in subcutaneous layer. The dermis layer has more variation in the population density of blood vessels. Due to these reasons small values of m and S have been assigned to the elements (9) and (21) which are situated at the interface of epidermis and dermis. The values of m and S are taken zero for all other elements which are entirely in the epidermis layer. Two different set of values of m and S have been considered for the elements of dermis layer. All elements of subcutaneous layer have same values of m and S. Constant but different values of thermal conductivity has been assigned to each layer. Detailed information about the elements is given in Tables II–IV. Nodal temperatures have been calculated for three sets of skin thickness shown in Table I.

TABLE I

Skin layers thickness in cms

Set	y_1	y_2	y_3	y_4	y_5	a
I	0.06	0.11	0.16	0.18	0.20	0.10
II	0.16	0.24	0.32	0.36	0.40	0.10
III	0.20	0.30	0.40	0.45	0.50	0.10

TABLE II

Element information for epidermis layer

Nodal coordinates			Element information						
Node	x_i	y_i	e	i	j	k	$K\times10^{-3}$	$mc_b\times10^{-3}$	$S\times10^{-3}$
5	0	y_3	9	5	6	12	0.50	0.11	0.29
6	0	y_4	10	6	12	13	0.50	0.00	0.00
7	0	y_5	11	6	13	14	0.50	0.00	0.00
12	$a/2$	y_3	12	6	7	14	0.50	0.00	0.00
13	$a/2$	y_4	21	12	19	20	0.50	0.11	0.29
14	$a/2$	y_5	22	12	13	20	0.50	0.00	0.00
19	a	y_3	23	13	14	20	0.50	0.00	0.00
20	a	y_4	24	14	20	21	0.50	0.00	0.00
21	a	y_5							

TABLE III

Element information for dermis layer

Nodal coordinates			Element information						
Node	x_i	y_i	e	i	j	k	$K\times10^{-3}$	$mc_b\times10^{-3}$	$S\times10^{-3}$
3	0	y_1	5	3	4	10	0.75	0.33	0.88
4	0	y_2	6	4	10	11	0.75	0.33	0.88
5	0	y_3	7	4	11	12	0.75	0.22	0.58
10	$a/2$	y_1	8	4	5	12	0.75	0.22	0.58
11	$a/2$	y_2	17	10	17	18	0.75	0.33	0.88
12	$a/2$	y_3	18	10	11	18	0.75	0.33	0.88
17	a	y_1	19	11	12	18	0.75	0.22	0.58
18	a	y_2	20	12	18	19	0.75	0.22	0.58
19	a	y_3							

TABLE IV

Element information for subcutaneous layer

Nodal coordinates			Element information						
Node	x_i	y_i	e	i	j	k	$K\times10^{-3}$	$mc_b\times10^{-3}$	$S\times10^{-3}$
1	0	0	1	1	2	8	1.0	0.44	1.17
2	0	$y_1/2$	2	2	8	9	1.0	0.44	1.17
3	0	y_1	3	2	9	10	1.0	0.44	1.17
8	$a/2$	0	4	2	3	10	1.0	0.44	1.17
9	$a/2$	$y_1/2$	13	8	15	16	1.0	0.44	1.17
10	$a/2$	y_1	14	8	9	16	1.0	0.44	1.17
15	a	0	15	9	10	16	1.0	0.44	1.17
16	a	$y_1/2$	16	10	16	17	1.0	0.44	1.17
17	a	y_1							

The values of thermal conductivity, blood mass flow rate and rate of metabolic heat generation, used for numerical calculation, have been taken from Chao *et al.* (1973). As has been stated earlier that in this study the skin surface is supposed to be exposed to cool and dry environment. Thus the effect of sweet evaporation is not accounted for the calculation of nodal temperatures. The nodal temperatures have been obtained for the skin surface temperature at 13°C, 15°C, 17°C and 19°C. The results are listed in Table V.

CONCLUSION

From the results it appears that nodal temperatures increase gradually from epidermis to subcutaneous layer.

For sets II and III the epidermis nodal temperatures are much higher than surface temperature but this difference is not significant in set I. It may be because epidermis is thinner in set I than sets II and III so it is easily effected by the surface temperature. In case of more thickness the nodal temperatures are generally effected by body core temperature.

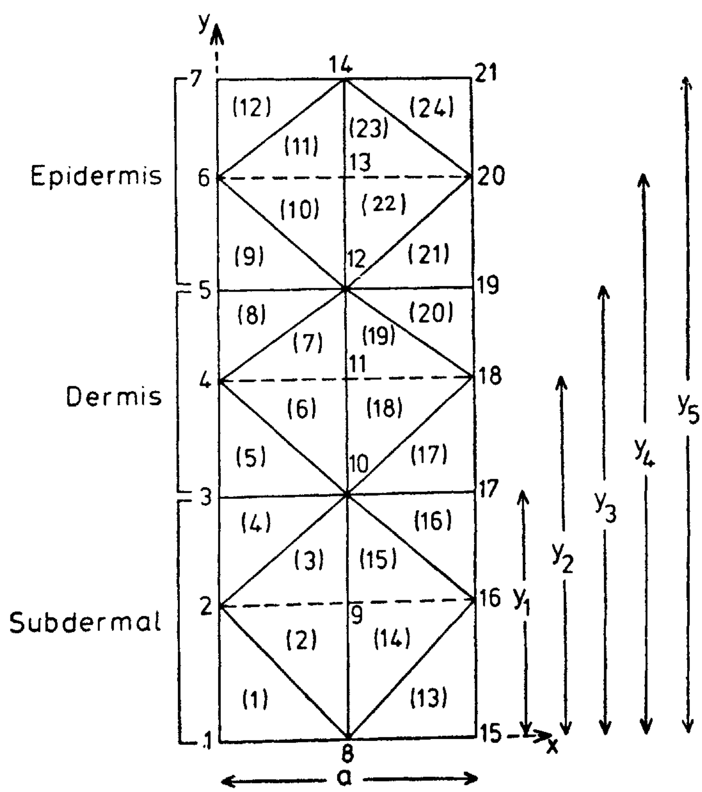

FIG. 2. Finite element arrangement for the whole region.

TABLE V

Nodal temperatures in degree centigrade

θ_0	θ_2	θ_3	θ_4	θ_5	θ_6	θ_9	θ_{10}	θ_{11}	θ_{12}	θ_{13}	θ_{16}	θ_{17}	θ_{18}	θ_{19}	θ_{20}
Set I															
13	27.07	21.39	16.67	14.71	13.85	21.17	17.27	16.16	14.62	13.82	27.12	21.46	16.93	14.79	13.88
15	27.22	21.73	17.46	16.04	15.52	21.30	17.59	16.96	15.94	15.48	27.28	21.80	17.73	16.12	15.55
17	27.38	22.07	18.26	17.36	17.18	21.42	17.90	17.76	17.27	17.14	27.44	22.15	18.54	17.45	17.21
19	27.54	22.41	19.05	18.69	18.84	21.54	18.22	18.56	18.60	18.80	27.59	22.49	19.35	18.78	18.88
Set II															
13	33.27	29.39	23.71	17.26	14.87	33.29	29.46	23.83	18.26	14.94	33.32	29.58	23.92	18.59	15.46
15	33.52	29.89	24.53	18.36	16.44	33.54	29.96	24.67	19.44	16.44	33.57	30.09	24.76	19.80	17.02
17	33.77	30.04	25.36	19.45	18.00	33.79	30.46	25.51	20.63	17.94	33.83	30.59	25.60	21.02	18.59
19	34.02	30.90	26.18	20.55	19.57	34.04	30.96	26.34	21.80	19.45	34.08	31.10	26.44	22.23	20.16
Set III															
13	33.66	33.18	25.41	20.52	16.76	33.69	30.19	25.41	20.52	16.76	33.58	30.16	25.41	20.52	16.76
15	33.96	30.79	26.42	21.92	18.46	33.99	30.80	26.41	21.92	18.46	33.87	30.76	26.41	21.92	18.46
17	34.26	31.39	27.42	23.32	20.16	34.30	31.40	27.41	23.32	20.16	34.18	31.37	27.41	23.31	20.16
19	34.56	32.00	28.42	24.71	21.86	34.60	32.00	28.41	24.71	21.86	34.48	31.98	28.41	24.71	21.86

$\theta_1 = \theta_8 = \theta_{15} = 37°C$ (Body core temperature), $\theta_7 = \theta_{14} = \theta_{21} = \theta_0$ (Skin surface temperature).

In sets II and III it is seen that the nodes situated at the same distance from the skin surface have almost equal temperature irrespective of layer to which they belong. This is not evident in set I. Also, there is much difference between the nodal temperatures of sets I, II and III. Again, these differences may be attributed to the more skin thickness in sets II and III than set I.

The population density of blood vessels may be more and uniform in thicker skins so due to more blood flow the nodal temperatures are much effected by body core temperature. This may be one of the reasons that nodes situated at same distance from the skin surface have almost same temperature. Other reason may be more fat deposition which prevents the dissipation of heat and acts as an insulator.

In all the cases the nodal temperatures of epidermis layer are more or less effected by the skin surface temperature while dermis and subcutaneous nodal temperatures are mainly influenced by body core temperature.

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