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ON UNSTEADY MHD FLOW THROUGH TWO PARALLEL POROUS FLAT PLATES

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An attempt is made to study the flow of an electrically conducting, incompressible, viscous fluid between two parallel porous flat plates under a time varying pressure gradient. By using Laplace transform technique an exact solution for the problem has been obtained when the pressure gradient is constant. The effect of suction and injection on the flow field has been studied. It has been shown that the transient part of the solution obtained decays more rapidly with increase in transverse magnetic field which results in early attainment of steady state.

INTRODUCTION

The problem of steady viscous flow in the annular space bounded by two concentric circular cylinders, when the inner wall is discharging fluid and the outer wall is absorbing it, has been studied by Berman (1958). Rosenhead (1963) has obtained an exact solution of the problem of steady viscous flow past a flat plate at zero incidence with uniform suction. Satya Prakash (1969) has studied the problem of unsteady incompressible viscous flow under a time varying pressure gradient in a straight channel with two parallel porous flat walls, when one wall is discharging fluid and another wall absorbing it. This paper is concerned with the unsteady flow of an electrically conducting, incompressible, viscous fluid between two parallel porous flat plates with suction and injection and is an extension of the work of Satya Prakash (1969) to MHD case. Here the magnetic Reynolds number is assumed very small so that the induced magnetic field can be neglected.

MATHEMATICAL FORMULATION

We consider the two dimensional unsteady flow of an electrically conducting, incompressible viscous fluid bounded by two parallel porous flat plates at $y = 0$ and $y = d$ in the presence of uniform transverse magnetic field B along y-axis. We take the axis as shown in Fig. 1.

We assume that the liquid is being injected into the flow region through the plate at $y = 0$ and is being sucked away through the plate at $y = d$. Let u and v be the components of the velocity at a point (x, y) of the flow region.

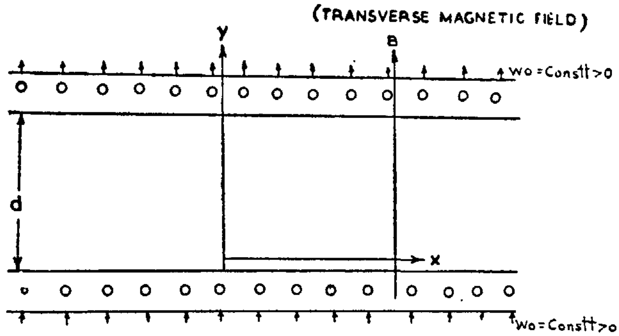

FIG. 1. Parallel porous flat plates with uniform injection through the plate at $y = 0$ and suction through the plate at $y = d$.

The unsteady motion of the electrically conducting incompressible viscous fluid is governed by the following equations of motion and the continuity equation in usual notation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{\sigma}{\rho}B_0^2 u \qquad \ldots(1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \qquad \ldots(2)$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad \ldots(3)$$

where ρ is the density of the liquid, ν the kinematic viscosity, t the time, p the pressure at a point whose coordinates are (x, y), σ the electrical conductivity and B the magnetic field.

Let the initial and boundary conditions be given by

$$t \leqslant 0,\ u = 0 \text{ and } v = 0 \text{ for } 0 \leqslant y \leqslant d \qquad \ldots(4)$$

$$t > 0,\ u = 0 \text{ and } v = w_0 = \text{positive constant for } y = 0 \text{ and } y = d. \qquad \ldots(5)$$

In view of conditions (4) and (5), it follows that the velocity field distribution is independent of x

$$\frac{\partial u}{\partial x} = 0,\ \frac{\partial v}{\partial x} = 0. \qquad \ldots(6)$$

Substituting $\frac{\partial u}{\partial x} = 0$ in eqn. (3) and using condition (5) we get $v = W_0$.

Now putting $v = w_0$, eqns. (1) and (2) become

$$\frac{\partial u}{\partial t} + w_0 \frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma}{\rho} B_0^2 u \qquad \ldots(7)$$

and

$$0 = \frac{\partial p}{\partial y}. \qquad \ldots(8)$$

Now with the help of assumption made by Rossow (1958), the non-linear term in eqn. (7) disappears automatically and the electromagnetic body force involved in eqn. (7) takes the linearized form

$$-\frac{\sigma}{\rho} B_0^2 u = -mu. \qquad \ldots(9)$$

With the help of eqn. (9), eqn. (7) becomes

$$\frac{\partial u}{\partial t} + w_0 \frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} - mu \qquad \ldots(10)$$

and

$$0 = \frac{\partial p}{\partial y}. \qquad \ldots(10a)$$

Let us put

$$u' = \frac{u}{w_0},\ x' = \frac{x}{d},\ y' = \frac{y}{d} \qquad \ldots(11)$$

$$p' = \frac{p}{\rho w_0^2},\ t' = \frac{t w_0}{d},\ M = B_0 \sqrt{\frac{\sigma d}{\rho w_0}} \text{ and } R = \frac{w_0 d}{\nu}$$

where w_0 is the characteristic velocity, d the characteristic length and ρw_0^2 and $\frac{d}{w_0}$ are characteristic pressure and time respectively.

By virtue of (11), eqns. (6), (10) and (10a) become

$$\frac{\partial u'}{\partial t'} + \frac{\partial u'}{\partial y'} = -\frac{\partial p'}{\partial x'} + \frac{1}{R}\frac{\partial^2 u'}{\partial y'^2} - M^2 u' \qquad \ldots(12)$$

$$0 = \frac{\partial p'}{\partial y'} \qquad \ldots(13)$$

in non-dimensional forms.

The initial and boundary conditions (4) and (5) become

$$t' \leqslant 0;\ u' = 0 \quad \text{for } 0 \leqslant y' \leqslant 1 \qquad \ldots(14)$$

$$t' > 0;\ u' = 0 \quad \text{for } y' = 0, \text{ and } y' = 1. \qquad \ldots(15)$$

Equation (13) shows that u' is independent of x', and therefore a function of y' and t' only. From eqn. (13) it is clear that p' is independent of y' and hence from eqn. (12) it follows that $\frac{\partial p'}{\partial x'}$ is a function of t' alone.

Taking

$$\frac{\partial p'}{\partial x'} = -f(t') \qquad \text{...(16)}$$

eqn. (12) yields

$$\frac{\partial u'}{\partial t'} + \frac{\partial u'}{\partial y'} = f(t') + \frac{1}{R}\frac{\partial^2 u'}{\partial y'^2} - M^2 u'. \qquad \text{...(17)}$$

Solution of the Problem

With the help of Laplace transformation, eqn. (17) reduces to

$$\int_0^\infty \frac{\partial u'}{\partial t'} e^{-\alpha t'} dt' + \int_0^\infty \frac{\partial u'}{\partial y'} e^{-\alpha t'} dt'$$

$$= \int_0^\infty f(t')\, e^{-\alpha t'} dt' + \frac{1}{R}\int_0^\infty \frac{\partial^2 u'}{\partial y'^2} e^{-\alpha t'} dt' - M^2 \int_0^\infty u'\, e^{-\alpha t'} dt' \qquad \text{...(18)}$$

or

$$u' e^{-\alpha t'} + \alpha \int_0^\infty u' e^{-\alpha t'} dt' + \int_0^\infty \frac{\partial u'}{\partial y'} e^{-\alpha t'} dt'$$

$$= \int_0^\infty f(t')\, e^{-\alpha t'} dt' + \frac{1}{R}\int_0^\infty \frac{\partial^2 u}{\partial y'^2} e^{-\alpha t'} dt' - M^2 \int_0^\infty u'\, e^{-\alpha t'} dt'. \qquad \text{...(19)}$$

By virtue of condition (14), the first term on the left of eqn. (19) vanishes and denoting the Laplace transforms of u' and $f(t')$ by

$$\bar{u}' = \int_0^\infty u' e^{-\alpha t'} dt' \qquad \text{...(20)}$$

and

$$\bar{f}(\alpha) = \int_0^\infty f(t')\, e^{-\alpha t'} dt' \qquad \text{...(21)}$$

eqn. (19) after simplification reduces to

$$\frac{d^2\bar{u}'}{dy'^2} - R\frac{d\bar{u}'}{dy'} - (R\alpha + M^2)\,\bar{u}' = -R\bar{f}(\alpha) \qquad ...(22)$$

The conditions (15) transform to

$$\bar{u}' = 0 \text{ for } y' = 0, 1. \qquad ...(23)$$

The solution of eqn. (22) satisfying the conditions (23) is

$$\bar{u}' = \frac{\bar{f}(\alpha)}{\alpha \sinh\left(\frac{1}{2}[R^2 + 4(\alpha R + M^2)]^{1/2}\right)}$$
$$\times \left(- e^{-(1-y')R/2} \sinh\left\{\frac{[R^2 + 4(\alpha R + M^2)]^{1/2}\, y'}{2}\right\}\right.$$
$$\left. - e^{Ry'/2} \sinh\left\{\frac{(R^2 + 4(\alpha R + M^2))^{1/2}}{2}(1 - y')\right\}\right) + \frac{\bar{f}(\alpha)}{\alpha}. \qquad ...(24)$$

With the help of inversion formula given by Tranter (1951), we get

$$u' = \frac{1}{2\pi i}\int_{\bar{y}-i\omega}^{\bar{y}+i\omega} \bar{u}' e^{\alpha t'}\, dt' \qquad ...(25)$$

where $\bar{y}$ is greater than real part of all the singularities of u'.

Special Case

Suppose the pressure gradient is a constant quantity that is

$$\frac{\partial p'}{\partial x'} = -P \qquad ...(26)$$

where P is a positive constant. Then

$$f(\alpha) = \int_0^\infty P e^{-\alpha t'}\, dt' = \frac{P}{\alpha}. \qquad ...(27)$$

Hence from eqns. (24) and (25), we have in this case

$$u' = \frac{1}{2\pi i}\int_{\bar{y}-i\omega}^{\bar{y}+i\omega} \left[\frac{P}{\alpha^2 \sinh\left(\frac{1}{2}[R^2 + 4(\alpha R + M^2)]^{1/2}\right)}\right.$$
$$\times \left(- e^{\{-(1-y')R/2\}} \sinh\left\{\frac{[R^2 + 4(\alpha R + M^2)]^{1/2}}{2}\, y'\right\} - \right.$$

(equation continued on p. 1377)

$$- e^{Ry'/2} \sinh\left\{\frac{[R^2 + 4(\alpha R + M^2)]^{1/2}}{2}(1 - y')\right\}\bigg) + \frac{P}{\alpha^2}\bigg]e^{\alpha t'}dt'. \quad \ldots(28)$$

The integrand has a double pole at $\alpha = 0$ and simple poles at

$$\alpha = -\frac{R^2 + 4\pi^2 n^2 + 4M^2}{4R} \quad (n = 0, 1, 2 \ldots)$$

and the residues at these poles are

$$\frac{P}{2\pi i}\left(y' - \frac{e^{Ry'} - 1}{e^R - 1}\right)$$

and

$$\frac{16PR_n e^{Ry'/2} \sin(n\pi y')\{e^{-R/2}(-1)^n - 1\} \exp\left(-\frac{(R^2+4\pi^2n^2+4M^2)\, t'}{4R}\right)}{i\,(R+4\pi^2n^2+4M^2)^2}$$

$$(n = 0, 1, 2, \ldots)$$

respectively.

By Carslaw and Jaeger (1941), we have

$$u' = P\left(y' - \frac{e^{Ry'} - 1}{e^R - 1}\right)$$

$$+ 32PR\pi \sum_{n=0}^{\infty} \frac{n e^{Ry'/2} \sin(n\pi y')\{e^{-R/2}(-1)^n - 1\} \exp\left(-\frac{(R^2+4\pi^2n^2+4M^2)t'}{4R}\right)}{(R^2 + 4\pi^2n^2 + 4M^2)^2}. \quad \ldots(29)$$

In the absence of magnetic field i.e. $M \to 0$, the solutions (28) and (29) are exactly identical with the solutions of Satya Prakash (1969).

Conclusion

From Figs. 2 and 3 it is evident that the velocity increases with time and tends ultimately towards the steady state at both the points. This conclusion is expected to remain valid for all points of the channel. The velocity at the point near the wall which is subjected to injection is less than that at the point near the wall which is subjected to suction. This is on account of the fact that injection introduces decelerated liquid particles into the flow and suction removes the retarded liquid particles from the flow. This fact was also noted by Satya Prakash (1969). It is remarkable here that the transient part of the solution given by the exponential part under series summation decays more rapidly with the increase in magnetic parameter M at all points of the flow field and therefore the steady state is reached faster with the increase in magnetic parameter M.

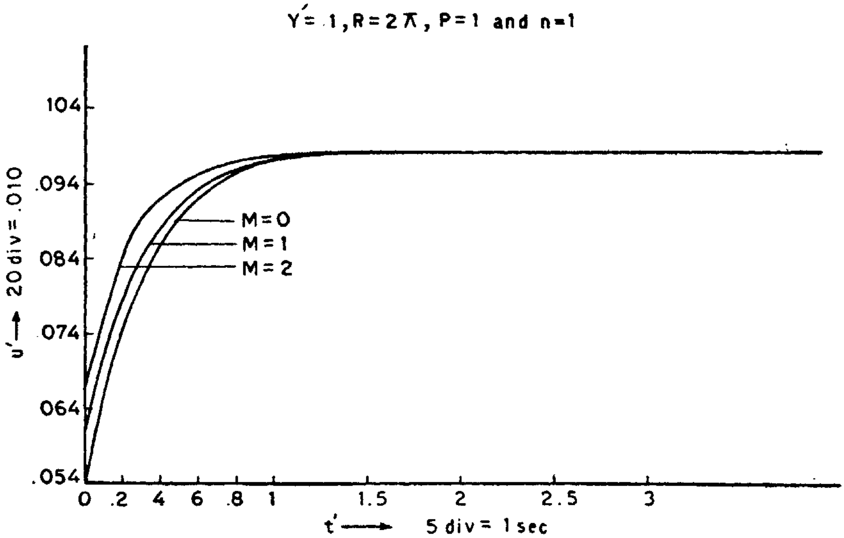

FIG. 2. Velocity variations with time for $M = 0$, 1 and 2, at $Y' = 0.1$.

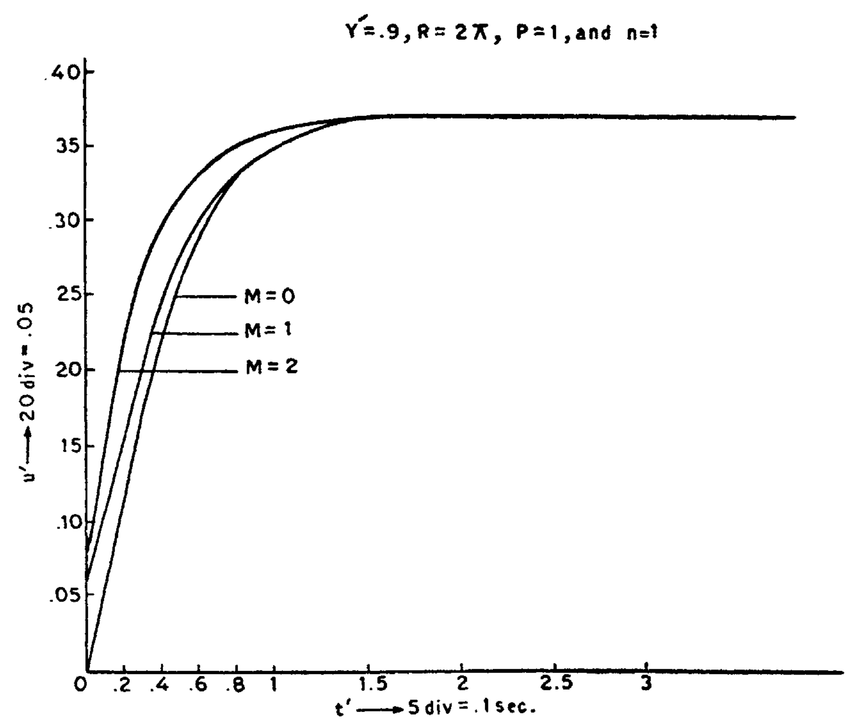

FIG. 3. Velocity variations with time for $M = 0$, 1 and 2 at $Y' = 0.9$.

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REFERENCES

Berman, A. S. (1958). Laminar flow in an annulus with porous walls. *J. appl. Phys.*, **29**, 71-75.

Carslaw, H. S., and Jaeger, J. C. (1941). Operational Methods in Applied Mathematics. Clarendon Press, Oxford, pp. 76–77.

Rosenhead, L. (1963). Laminar Boundary Layers. Clarendon Press, Oxford, p. 141.

Rossow, V. J. (1958). On the flow of electrically conducting fluids over a flat plate in the presence of a transverse magnetic field. *Nasa Report*, pp. 489–508.

Satya Prakash (1969). An exact solution of the problem of unsteady viscous flow through porous straight channel. *Proc. natn. Inst. Sci. India* **35** A, 123-29.

Tranter, C. J. (1951). Integral Transforms in Mathematical Physics. Mathwen & Co. Ltd., London, p. 15.