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NONLINEAR ANALYSIS OF HEATED ORTHOTROPIC PLATES

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Governing equations for the large thermal deflections of orthotropic plates have been derived. For a rectangular plate with immovable edges and subjected to a steady temperature distribution through the thickness, the solution for the stress function is presented and using Galerkin's procedure, a cubic equation for the deflection has been obtained in the nondimensional form. The numerical results for the deflections and membrane stresses have also been presented.

NOTATIONS

$[a]$ = matrix of material constants in equation for strains in terms of stresses,

S_{ij} = elements of $[a]$ in eqn. (2),

(E) = matrix of material constants in equation for stresses in terms of strains, $[E] = [a]^{-1}$,

α_1, α_2 = co-efficients of thermal expansion in x and y directions respectively,

ϵ_1, ϵ_2 = normal strains in the x and y directions respectively,

γ_{12} = shear strain,

σ_x, σ_y = normal stresses in the x and y directions respectively,

σ_{xy} = shear stress,

F = stress function,

u, v, w = displacements in the x, y and z directions,

x, y = independent in-plane variables,

$T(x, y, z)$ = temperature distribution in the plate,

α_t = co-efficient of thermal expansion (for isotropy),

h = plate thickness,

$K_1, K_2\ K_3$ = thermal conductivities,

∇^2 = Laplacian operator,

ν_1, ν_2 = Poisson's ratio (orthotropy),

ν = Poisson's ratio (isotropy).

INTRODUCTION

In high speed aircrafts, turbine and machine structures and in the fields of chemical and nuclear engineering there are numerous problems in which thermal stresses play an important and even a primary role. Compared to isotropic cases relatively few investigations of large thermal deflections and stresses for orthotropic plates have been made. This is due in part to the mathematical complexity of such problems and also to the fact that engineering structures have generally been fabricated from materials which are essentially isotropic. However, owing to the increased usage of anisotropic construction materials (e.g., fibre-reinforced composites) in situations involving severe thermal environments, there is an obvious need for further research in this area.

To the author's knowledge large deflection of heated orthotropic plates was first treated by Pal (1973) and subsequently by the present author (Biswas 1978) using Berger's (1955) approximation. Attempt was, however, made by Banerjee (1979) to derive coupled non-linear equations for large thermal deflections of orthotropic plates, but these governing equations and the boundary conditions for the in-plane displacements seem to be fallacious. In that paper no solution for the stress function nor any numerical results for deflections and stresses were presented.

The present paper is concerned with the derivation of governing equations for the large deflections of heated orthotropic plates. These equations exactly lead to von Karman's coupled non-linear equations for heated isotropic plates (Nowacki 1962). For a rectangular plate under stationary temperature distribution through the thickness of the plate, stress function has been derived and using Galerkin's procedure, a cubic equation for the deflection has been obtained.

Numerical results for the deflections and membrane stresses have been presented.

DERIVATION OF BASIC EQUATIONS

The orthotropic stress-strain relations are

$$(\epsilon) = (a)(\sigma) + (\alpha)T \qquad \ldots(1)$$

where

$$(a) = \begin{pmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{pmatrix}, \quad (\alpha) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{pmatrix}$$

$$(\epsilon) = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{pmatrix}, \quad (\sigma) = \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix}. \qquad \ldots(2)$$

The inverse of eqn. (1) is

$$(\sigma) = (E)(\epsilon) - (E)(\alpha)T. \quad ...(3)$$

The von Karman strain-displacement equations are

$$\epsilon_1 = \frac{\partial u}{\partial x} + \tfrac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z\,\frac{\partial^2 w}{\partial x^2}$$

$$\epsilon_2 = \frac{\partial v}{\partial y} + \tfrac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - z\,\frac{\partial^2 w}{\partial y^2}$$

$$\gamma_{12} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y} - 2z\,\frac{\partial^2 w}{\partial x\,\partial y}. \quad ...(4)$$

The forces N_x, N_y, N_{xy} are given by the expression

$$(N_x, N_y, N_{xy}) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \sigma_{xy})\, dz \quad ...(5)$$

and the moments M_x, M_y, and M_{xy} are given by

$$(M_x, M_y, M_{xy}) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \sigma_{xy})\, z dz. \quad ...(6)$$

The in-plane equations of equilibrium in the x and y directions are

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0, \quad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0. \quad ...(7)$$

These equations are identically satisfied by introducing the Airy stress function defined by

$$N_x = \frac{\partial^2 F}{\partial y^2};\ N_y = \frac{\partial^2 F}{\partial x^2};\ N_{xy} = -\frac{\partial^2 F}{\partial x\,\partial y}\ . \quad ...(8)$$

Solving for the stresses one gets from (3)

$$\left.\begin{aligned} \sigma_x &= \frac{S_{22}}{\Delta}\,\epsilon_1 - \frac{S_{12}}{\Delta}\,\epsilon_2 - \beta_1 T \\ \sigma_y &= \frac{S_{11}}{\Delta}\,\epsilon_2 - \frac{S_{12}}{\Delta}\,\epsilon_1 - \beta_2 T \\ \sigma_{xy} &= \gamma_{12}/S_{66} \end{aligned}\right\} \quad ...(9)$$

where

$$\beta_1 = \frac{\alpha_1 S_{22} - \alpha_2 S_{12}}{\Delta},\ \beta_2 = \frac{\alpha_2 S_{11} - \alpha_1 S_{12}}{\Delta}\ . \quad ...(10)$$

and $\Delta = S_{11}S_{22} - S_{12}^2$.

Combining eqns. (3), (8) and (9) one gets

$$\frac{\partial^2 F}{\partial y^2} = N_x = \frac{S_{22}}{\Delta} h \left(\epsilon_1 + z\frac{\partial^2 w}{\partial x^2}\right) - \frac{S_{12}}{\Delta} h \left(\epsilon_2 + z\frac{\partial^2 w}{\partial y^2}\right) - \beta_1 N_T$$

$$\frac{\partial^2 F}{\partial x^2} = N_y = \frac{S_{11}}{\Delta} h \left(\epsilon_2 + z\frac{\partial^2 w}{\partial y^2}\right) - \frac{S_{12}}{\Delta} h \left(\epsilon_1 + z\frac{\partial^2 w}{\partial x^2}\right) - \beta_2 N_T$$

$$\frac{\partial^2 F}{\partial x \partial y} = -N_{xy} = -\frac{1}{S_{66}} h \left(\gamma_{12} + 2z\frac{\partial^2 w}{\partial x \partial y}\right) \qquad \ldots(11)$$

where $N_T = \int_{-h/2}^{h/2} T\, dz.$

Taking the second derivatives of eqn. (4) it can be shown that

$$\frac{\partial^2 \epsilon_1}{\partial y^2} + \frac{\partial^2 \epsilon_2}{\partial x^2} - \frac{\partial^2 \gamma_{12}}{\partial x \partial y} = w_{xy}^2 - w_{xx}w_{yy} \qquad \ldots(12)$$

which is the compatibility equation.

Solving for the expressions for ϵ_1, ϵ_2, and γ_{12} given by equations (11) and putting the same into eqn. (12) one gets*

$$\frac{\partial^4 F}{\partial x^4} + p^2 \frac{\partial^4 F}{\partial x^2 \partial y^2} + q^2 \frac{\partial^4 F}{\partial y^4} + \lambda_1 \frac{\partial^2 N_T}{\partial x^2} + \lambda_2 \frac{\partial^2 N_T}{\partial y^2}$$

$$= E_2 h\,(w_{xy}^2 - w_{xx}w_{yy}) \qquad \ldots(13)$$

where

$$\left.\begin{aligned}
p^2 &= \frac{S_{66} + 2S_{12}}{S_{22}} = \frac{E_2}{G} - 2\nu_2, \quad q^2 = S_{11}/S_{22} = E_2/E_1 \\
\lambda_1 &= \frac{S_{11}\beta_1 + S_{12}\beta_2}{S_{22}} = \frac{E_2}{E_1}\beta_1 - \nu_2\beta_2 \\
\lambda_2 &= \frac{S_{12}\beta_1 + S_{22}\beta_2}{S_{22}} = \beta_2 - \nu_2\beta_1 \\
E_1 &= 1/S_{11}, \ E_2 = 1/S_{22}, \ S_{12}/S_{11} = -\nu_1, \ S_{12}/S_{22} = -\nu_2 \\
G &= 1/S_{66}.
\end{aligned}\right\} \qquad \ldots(14)$$

Combining eqns. (6) and (7) one gets

*The equation deduced by Banerjee (1979) appears to be fallacious. The equation deduced by him is

$$\frac{\partial^4 F}{\partial x^4} + p^2 \frac{\partial^4 F}{\partial x^2 \partial y^2} + q^2 \frac{\partial^4 F}{\partial y^4} + \frac{\alpha_1}{hS_{22}} \frac{\partial^2 T}{\partial x^2} + \frac{\alpha_2}{hS_{22}} \frac{\partial^2 T}{\partial y^2} = E_2(w_{xy}^2 - w_{xx}w_{yy})$$

which is quite different from eqn. (13) of this paper.

$$\left.\begin{aligned} M_x &= -(D_x w_{xx} + D_1 w_{yy} + \beta_1 M_T) \\ M_y &= -(D_y w_{yy} + D_1 w_{xx} + \beta_2 M_T) \\ M_{xy} &= 2D_{xy} w_{xy} \end{aligned}\right\} \quad \ldots(15)$$

where $$M_T = \int_{-h/2}^{h/2} z\,T dz$$

$$\left.\begin{aligned} D_x &= E'_x h^3/12,\ E'_x = S_{22}/\Delta,\ D_{xy} = \frac{Gh^3}{12},\ G = \frac{1}{S_{66}} \\ D_y &= E'_y h^3/12,\ E'_y = S_{11}/\Delta \\ D_1 &= E'' h^3/12,\ E'' = -S_{12}/\Delta. \end{aligned}\right\} \quad \ldots(16)$$

Putting the expressions (15) into the equation of equilibrium (Timoshenko and Krieger 1959, p. 379)

$$\frac{\partial^2 M_x}{\partial x^2} - \frac{2\partial^2 M_{xy}}{\partial x\,\partial y} + \frac{\partial^2 M_y}{\partial y^2} = -(N_{xx} w_{yy} + 2N_{xy} w_{xy} + N_{yy} w_{xx})$$

one gets

$$D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial y^2 \partial x^2} + D_y \frac{\partial^4 w}{\partial y^4} + \beta_1 \frac{\partial^2 M_T}{\partial x^2} + \beta_2 \frac{\partial^2 M_T}{\partial y^2}$$
$$= F_{xx} w_{yy} - 2F_{xy} w_{xy} + F_{yy} w_{xx}. \quad \ldots(17)$$

The coupled equations (15) and (17) will be used for determining the stress function and deflection of heated orthotropic plates. For isotropy these two equations lead to

$$\nabla^4 F = Eh(w_{xy}^2 - w_{xx} w_{yy}) - \alpha_t Eh(\nabla^2 \tau_0)$$

$$D\nabla^4 w = -\frac{1}{12(1-\nu)} \alpha_t Eh^3(\nabla^2 \tau) + F_{xx} w_{yy} - 2F_{xy} w_{xy} + F_{yy} w_{xx} \quad \ldots(18)$$

which are von Karman classical non-linear equations extended to thermal loading (Nowacki 1962, p. 490).

SOLUTION FOR A RECTANGULAR PLATE

We consider a simply-supported rectangular plate occupying the space

$$0 \leqslant x \leqslant a;\ 0 \leqslant y \leqslant b;\ -\tfrac{1}{2}h \leqslant z \leqslant \tfrac{1}{2}h. \quad \ldots(19)$$

The plate is subjected to a steady state temperature distribution which satisfies the Fourier heat-conduction equation

$$K_1 \frac{\partial^2 T}{\partial x^2} + K_2 \frac{\partial^2 T}{\partial y^2} + K_3 \frac{\partial^2 T}{\partial z^2} = 0. \quad \ldots(20)$$

If the temperature varies linearly through the thickness, then one can assume

$$T(x, y, z) = \frac{T_1 + T_2}{2} + z\left(\frac{T_1 - T_2}{h}\right) \qquad \text{...(21)}$$

compatible with the boundary conditions

$$T(x, y, \tfrac{1}{2}h) = T_1 \text{ and } T(x, y, -\tfrac{1}{2}h) = T_2. \qquad \text{...(22)}$$

We assume the deflection w satisfying the conditions of simply-supported edges in the form

$$w = w_0 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}. \qquad \text{...(23)}$$

Since N_T is constant, the general solution of eqn. (13) after substituting eqn. (8), is obtained in the form

$$F = A\frac{x^2}{2} + B\frac{y^2}{2} + \frac{1}{32} E_2 h w_0^2 \left(\frac{a^2}{b^2}\cos\frac{2\pi x}{a} + \frac{b^2}{a^2 q^2}\cos\frac{2\pi y}{b}\right) \qquad \text{...(24)}$$

where A and B are arbitrary constants to be determined from in-plane boundary conditions.

In accordance with the conditions occurring in airplane structures the plate is considered rigidly framed, all edges thus remaining straight after deformation. Then the elongations of the plate in the directions of x and y are independent of y and x respectively (Timoshenko and Krieger 1959, p. 426).

From eqns. (4) and (11) one gets*

$$u_x = \frac{S_{11}F_{yy} + S_{12}F_{xx}}{h} - \tfrac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\beta_1 S_{11} + \beta_2 S_{12}}{h} N_T \qquad \text{...(25)}$$

$$v_y = \frac{S_{22}F_{xx} + S_{12}F_{yy}}{h} - \tfrac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{\beta_1 S_{12} + \beta_2 S_{22}}{h} N_T. \qquad \text{...(26)}$$

From eqns. (25) and (26) and considering the statement cited above one gets

$$u = \int_0^a \left[\frac{S_{11}}{h}\left(B - \frac{E_2 h w_0^2 \pi^2}{8a^2q^2}\cos\frac{2\pi y}{b}\right) + \frac{S_{12}}{h}\left(A - \frac{E_2 h w_0^2 \pi^2}{8b^2}\cos\frac{2\pi x}{a}\right) - \tfrac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\beta_1 S_{11} + \beta_2 S_{12}}{h} N_T\right]_{y=0} dx \qquad \text{...(27)}$$

$$v = \int_0^b \left[\frac{S_{22}}{h}\left(A - \frac{E_2 h w_0^2 \pi^2}{8b^2}\cos\frac{2\pi x}{a}\right) + \frac{S_{12}}{h}\left(B - \frac{E_2 h w_0^2 \pi^2}{8a^2q^2}\cos\frac{2\pi y}{b}\right) - \tfrac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{\beta_1 S_{12} + \beta_2 S_{22}}{h} N_T\right]_{x=0} dy. \qquad \text{...(28)}$$

*The expressions for u_x and v_y deduced by Banerjee (1979) seem to be fallacious. The correct forms of those expressions are given by eqns. (25) and (26) of this paper.

For immovable edges of the plate

$$u = 0 = v \quad \text{(Nowacki 1962, p. 428).} \qquad \ldots(29)$$

From eqns. (27) and (28) and considering (29) one gets the constants A and B in the forms

$$A = C_1 w_0^2 - \beta_2 N_T \qquad \ldots(30)$$

$$B = C_2 w_0^2 - \beta_1 N_T \qquad \ldots(31)$$

where

$$C_1 = \frac{E_2 h \pi^2}{8(S_{12}^2 - S_{11}S_{22})} \left(\frac{S_{11}S_{22}}{a^2 q^2} - \frac{S_{22}S_{11}}{b^2} \right) \qquad \ldots(32)$$

$$C_2 = \frac{E_2 h \pi^2}{8(S_{11}S_{22} - S_{12}^2)} \left(\frac{S_{11}S_{22}}{a^2 q^2} - \frac{S_{22}S_{12}}{b^2} \right). \qquad \ldots(33)$$

Since M_T is constant, it can be expanded in the form of Fourier series

$$M_T = \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \frac{16\, M_T}{mn\pi^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}. \qquad \ldots(34)$$

Putting the expressions (23), (24) and (34) into eqn. (17) and applying Galerkin's procedure one gets the following cubic equation determining (w_0/h).

$$\left(\frac{w_0}{h}\right)^3 \left[1 + \left(\frac{b}{a}\right)^4 \left(\frac{E_1}{E_2}\right)^2 - \frac{2E''}{\nu_1 E_2} \left\{\left(\frac{b}{a}\right)^2 \left(\frac{E_1}{E_2}\right)^2 - 1\right\} + 2\left\{\left(\frac{b}{a}\right)^4 \left(\frac{E_1}{E_2}\right)^2 \frac{1}{\nu_1} + \left(\frac{b}{a}\right)^2\right\} \frac{E''}{E_2}\right]$$
$$+ \frac{w_0}{h}\left[\frac{4}{3}\,\frac{E_1/E_2(b/a)^4}{1-\nu_1\nu_2} + \frac{8}{3}\left(\frac{b}{a}\right)^2 \left\{\frac{E''/E_2}{1-\nu_1\nu_2} + 2G/E_2\right\} + \frac{4}{3(1-\nu_1\nu_2)} - \frac{16 b^2 \beta_2 N_T}{\pi^2 h^3 E_2}\left\{1 + \frac{\beta_1}{\beta_2}\left(\frac{b}{a}\right)^2\right\}\right]$$
$$- \frac{256\, M_T}{\pi^4 E_2}\left(\frac{b}{h}\right)^4 \frac{\beta_2}{b^2}\left\{1 + \frac{\beta_1}{\beta_2}\left(\frac{b}{a}\right)^2\right\} = 0. \qquad \ldots(35)$$

Dropping the non-linear term $(w_0/h)^3$ as well as N_T the corresponding thermal deflection of a rectangular plate is obtained as

$$\frac{w_0}{h} \left(= w_{max}/h\right) = \frac{16\,(M_T/h)\,(\beta_1 a^{-2} + \beta_2 b^{-2})}{\pi^4[(D_x/a^4) + (2H/a^2b^2) + (D_y/b^4)]} \qquad \ldots(36)$$

which is in agreement with the result in Sarkar (1967) as a first term approximation.

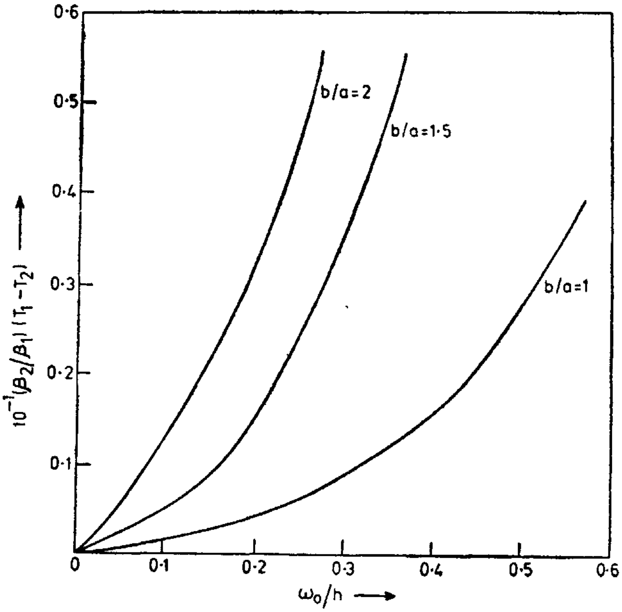

FIG. 1. Variation of non-dimensional deflections w_0/h for different values of the temperature parameter $\frac{10^{-1}\beta_2}{\beta_1}(T_1-T_2)$.

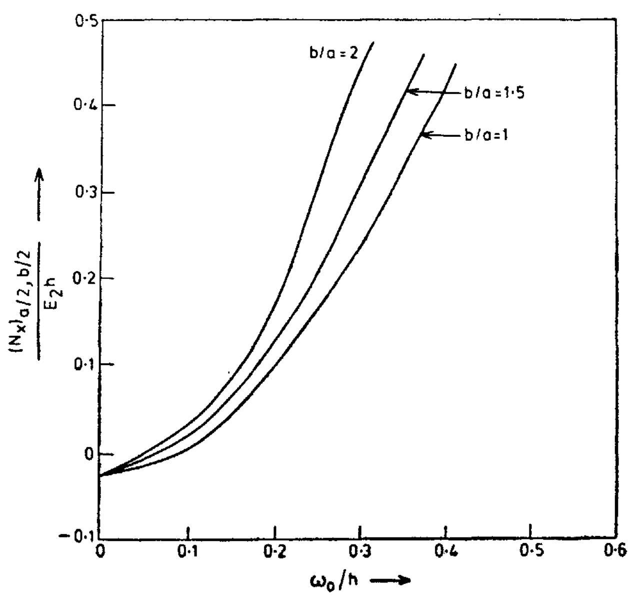

FIG. 2. Variation of non-dimensional membrane stresses $\frac{(N_x)_{a/2,b/2}}{E_2h}$ for different values of w_0/h.

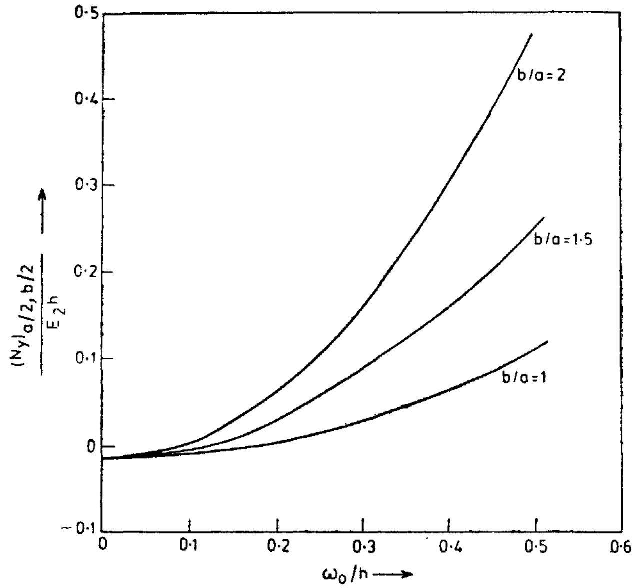

FIG. 3. Variation of non-dimensional stresses $\frac{(N_y)_{a/2,b/2}}{E_2 h}$ for different values of w_0/h.

NUMERICAL RESULTS

For different aspect ratios $\left(\frac{b}{a}\right)$ the variation of the non-dimensional deflection w_0/h for different values of $10^{-1}\,\frac{\beta_1}{E_2}\,(T_1 - T_2)$ have been presented in Fig. 1 considering the following set of values:

$$E_1 = 1 \times 10^5,\ E_2 = 0.05 \times 10^5,\ \nu_1 = 0.2,$$

$$\nu_2 = 0.01,\ G_{12} = 0.05 \times 10^5,\ \frac{b}{h} = 10,$$

$$\frac{\beta_2(T_1 + T_2)}{E_2} = 0.2,\ \frac{\beta_1}{\beta_2} = 0.19,\quad \frac{\alpha_2}{\alpha_1} = 0.8.$$

Also combining eqns. (11) and (24) membrane stresses $(N_x)_{max}$ and $(N_y)_{max}$ which occur at $x = a/2, y = b/2$ have been computed for different values of w_0/h and presented in Fig. 2.

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