

## THE DISPERSION RELATION FOR THE PROPAGATION OF HIGH FREQUENCY THERMAL WAVES IN DIELECTRICS

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The dispersion relation is obtained by using a model operator which differs from the one used in an earlier treatment (Mikhail and Simons 1975b). A comparison is then made between the present results and those obtained hitherto, in the cases of second sound and diffusive flow. A simple method is, finally, presented for the calculation of the normal processes relaxation time from measurements of the thermal wave attenuation and phase shift.

### 1. INTRODUCTION

A detailed study of the propagation of high frequency thermal waves in dielectrics was given by Mikhail and Simons (1975a, b, 1976). In the general case when both resistive and normal interactions are taken into consideration two alternative models were introduced to represent the phonon scattering mechanism (Mikhail and Simons 1975b). For the first model, the energy was supposed to be conserved only by the total collision operator while for the second it was assumed that each of the resistive and normal collision operators conserves energy separately. The first model was found to be mathematically simpler and was thus used throughout in (Mikhail and Simons 1975b) to derive the dispersion relation (the relation between the wave-number and frequency of the propagated thermal wave). From the physical point of view, however, it is not clear *a priori* which model is preferable\*. It is, therefore, of interest to reconsider the problem utilizing now the second model and to study the consequent variation which will occur in the dispersion relation. This is, in fact, the main purpose of the present work.

In §2 the second model is used to derive the dispersion relation which is found to take a form different from that obtained in Mikhail and Simons (1975b). The same series expansion technique previously used by Mikhail and Simons (1975a, b) is, then, employed in §3 to find explicit approximate formulae for the dispersion relation in the cases of diffusive flow and second sound. For second sound the result is identical

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\*Some authors (Guyer and Krumhansl 1964) used to describe the second approach to be more general but without giving any reasonable argument.

with that obtained by using the first model (Mikhail and Simons 1975b) while in the case of diffusive flow the present treatment gives rise to an extra term in the expression obtained for the first order deviation from the usual classical result. It is, however, shown in §4 that for both models this deviation can be put in a unique form which depends on the parameters specifying the normal processes relaxation time. This suggests that valuable information concerning the normal processes relaxation time might be obtained from experimental measurements of the thermal wave attenuation and phase shift.

### 2. THE DISPERSION RELATION

According to Mikhail and Simons (1975a) the Boltzmann equation describing the propagation of a thermal wave of frequency  $\omega$  and wavenumber  $\mathbf{p}$  in a dielectric crystal can be expressed in the form

$$i\omega\phi = L\phi - i\mathbf{p} \cdot \mathbf{v}\phi, \tag{1}$$

where  $\mathbf{v}$  is the sound velocity,  $\phi$  is a measure of the deviation of the phonon distribution function from the Bose Einstein equilibrium distribution ( $N_0$ ) and  $L$  is a linearized phonon collision operator. As was previously mentioned  $L$  will be replaced in the present treatment by the second model of Mikhail and Simons (1975b) which is given by

$$L_M\phi = HE\mu + FE\mathbf{v} + Gk_1\mathbf{v} - \eta\phi, \tag{2}$$

where

$$H = \frac{\langle E\mu\phi \rangle}{\langle E^2\mu \rangle}, F = \frac{\langle E\mathbf{v}\phi \rangle}{\langle E^2\mathbf{v} \rangle}, G = \frac{\langle k_1\mathbf{v}\phi \rangle}{\langle k_1^2\mathbf{v} \rangle} \tag{3}$$

and

$$\langle \psi \rangle = \int_{B.Z.} \left( -\frac{\partial N_0}{\partial E} \right) \psi \, d\mathbf{k}$$

$\mu$ ,  $\mathbf{v}$  and  $\eta$  are respectively the reciprocal of the resistive, normal and total relaxation times.  $E$  and  $\mathbf{k}$  are the phonon energy and wavenumber. The subscript  $M$  is used to distinguish between the exact and model operators while the subscript 1 stands for the component along the  $\mathbf{p}$  direction. Substitution of (2) into (1) yields

$$\phi = HE\mu\delta^{-1} + FE\mathbf{v}\delta^{-1} + Gk_1\mathbf{v}\delta^{-1} \tag{4}$$

where

$$\delta = \eta + i(\omega + p\mathbf{v}_1). \tag{5}$$

We, then, take the scalar product of both sides of (4) with  $E\mu$ ,  $E\mathbf{v}$  and  $k_1\mathbf{v}$ . The dispersion relation can be obtained by utilizing the fact that the determinant of the resulting

three linear homogeneous equations must vanish. It can be shown after some manipulation that the dispersion relation may be written in the form

$$D - \langle E^2 \mu \nu / \eta \rangle^{-1} S = 0 \tag{6}$$

where

$$D = i \langle E^2 \eta (\omega + p v_1) \delta^{-1} \rangle \langle k_1^2 \nu \Delta \delta^{-1} \rangle - \langle k_1 E \nu \eta \delta^{-1} \rangle^2,$$

$$S = \begin{vmatrix} i \langle E^2 \nu (\omega + p v_1) \delta^{-1} \rangle & -i \langle E^2 \eta (\omega + p v_1) \delta^{-1} \rangle & \langle E k_1 \nu \eta \delta^{-1} \rangle \\ i \langle E^2 \nu^2 \eta^{-1} (\omega + p v_1) \delta^{-1} \rangle & -i \langle E^2 \nu (\omega + p v_1) \delta^{-1} \rangle & \langle E k_1 \nu^2 \delta^{-1} \rangle \\ - \langle E k_1 \nu^2 \delta^{-1} \rangle & \langle E k_1 \nu \eta \delta^{-1} \rangle & - \langle k_1^2 \nu \Delta \delta^{-1} \rangle \end{vmatrix} \tag{7}$$

and

$$\Delta = \mu + i(\omega + p v_1).$$

The result obtained by using the first model [eqn. (8) in Mikhail and Simons (1975b)] is given by

$$D = 0. \tag{8}$$

It is, therefore, clear that the two results are identical if  $S = 0$ . It is readily shown that this is fulfilled in the following cases:

- (i)  $\mu = 0$ , the case of pure second sound.
- (ii)  $\nu = 0$ , the case of purely resistive interactions.
- (iii)  $\mu$  and  $\nu$  have the same  $k$  dependence, the case of normal and dislocation processes.

This is expected since in these cases the two models are equivalent.

### 3. APPROXIMATE SOLUTIONS

In order to solve eqn. (6) (to obtain the explicit relation between  $p$  and  $\omega$ ) either numerical or approximate analytical methods must be adopted. For small values of  $\omega$  it is convenient to use the series expansion technique originally presented by Mikhail and Simons (1975a, b). For this purpose  $\mu$ ,  $\nu$  and  $\eta$  can be adequately taken as

$$\left. \begin{aligned} \nu &= nx, \quad \mu = \sum_{r=1}^4 a_r x^r, \quad \eta = sx [1 + f(x)], \\ x &= \hbar \nu k / k_B T, \quad s = n + a_1, \quad f(x) = \sum_{r=2}^4 b_r x^{r-1} \\ b_r &= a_r / s, \quad r = 1, 2, 3, 4. \end{aligned} \right\} \tag{9}$$

where

and

On substituting, therefore, in (6) and using eqns. (13) and (A 3.14) in Mikhail and Simons (1975b) it can be shown after a fair amount of lengthy calculations that the dispersion relation can be expressed in the approximate form

$$D' - (3\mathcal{I})^{-1} S' = 0 \tag{10}$$

where  $D'$  is the L.H.S. of eqn. (14) in Mikhail and Simons (1975b),

$$S' = \begin{vmatrix} i\psi K_{4,1} + \frac{W^2}{3} K_{3,2} & i\psi F_4 + \frac{W^2}{3} K_{3,1} & \frac{1}{3} iWK_{4,1} \\ i\psi K_{4,2} + \frac{W^2}{3} K_{3,3} & i\psi K_{4,1} + \frac{W^2}{3} K_{3,2} & \frac{1}{3} iWK_{4,2} \\ iWK_{4,2} & iWK_{4,1} & \mathcal{I} \end{vmatrix}, \tag{11}$$

$\psi = \omega/s$ ,  $W = \rho v/s$  and the same notations as in Mikhail and Simons are used.

We now proceed to consider the cases of diffusive flow and second sound.

### 3.1. Diffusive Flow

In this case it is well known from the classical result that  $W^2 \propto \psi$  at low frequencies and thus it can be shown from (10) that the relation between  $W^2$  and  $\psi$  is given up to the second approximation by

$$W^2 = A\psi + [B(\psi) - C] \psi^2 \tag{12}$$

where  $A$  and  $B$  are defined by eqns. (17a, b) in Mikhail and Simons (1975b) and

$$C = \frac{iA}{\mathcal{I}} F_4 \left[ \frac{K_{4,1}}{F_4} - \frac{\mathcal{I}K_{3,2} + K_{4,2}K_{4,1}}{\mathcal{I}K_{3,1} + K_{4,1}^2} \right]^2. \tag{13}$$

The above form differs from the earlier result obtained in Mikhail and Simons (1975b) by the term  $C\psi^2$ . In view of what has been previously mentioned at the end of the preceding section it is clear that the contribution from this term is negligibly small at low temperatures where the dominant type of resistive interactions is dislocation scatter and at very high temperatures for which the strength of normal processes is substantially less than that of Umklapp processes. One might expect that this term would have its greatest effect in the vicinity of the temperature at which the thermal conductivity is maximum, since in this temperature range normal and resistive interactions are of the same order. It is, however, found that  $C/B$  never exceeds 1% for all the specimens of LiF considered by Berman and Brock (1965), for frequencies less than 300 kHz and temperatures less than 25 K.

### 3.2. Second Sound

In this case the phonon scattering mechanism is dominated by normal processes so that we can take  $\eta \sim v$  over the whole effective range of integration (the coefficients  $b_r$  are so small such that  $f(x_m) \ll 1$ , where  $x_m$  is the maximum effective upper limit

of all integrals). It is, therefore, clear from equation (7) that  $S \sim 0$  in this case and thus the same result obtained in Mikhail and Simons (1975b) will, finally, be deduced.

#### 4. DISCUSSION AND CONCLUSION

Although the two models yield different results in the case of diffusive flow it can still be shown that both results can be put into the form

$$p^2 = -ia\omega + \frac{a^3v^4}{10F_4s^3} \omega^3 \ln(-ig\omega) \quad \dots(14)$$

where  $a = c/\kappa$ ,

$c$  is the specific heat,  $\kappa$  is the thermal conductivity and  $g$  is a positive real constant which differs from one model to the other but can be expressed in both cases in terms of the integrals  $F_j$  and  $K_{j,r}$ . Also, if the leading terms are only considered then the real and imaginary parts of  $p$  are given respectively by

$$p_r = \sqrt{\frac{a\omega}{2}} + b\omega^{3/2} \left( \ln g\omega + \frac{\pi}{2} \right) \quad \dots(15a)$$

and

$$p_i = -\sqrt{\frac{a\omega}{2}} + b\omega^{3/2} \left( \ln g\omega - \frac{\pi}{2} \right) \quad \dots(15b)$$

where  $b = (a/2)^{5/2} (v^4/5F_4s^3)$ .

The first term in eqns. (14) and (15) is identical with the classical solution while the second term of these equations measures the first order deviation from this classical result. This indicates that the parameter  $s$  can be calculated from the gradient of the straight line representing the relation between the first order deviation and  $\omega^{3/2} \ln \omega$ . Also, if the temperature  $T$  is sufficiently high such that dislocation scattering can be neglected then  $s \sim n$  and the parameter specifying the normal processes relaxation time can thus be obtained. This, in fact, strongly confirms what was originally pointed out by Simons (1971) that the normal processes relaxation time can be obtained from measurements of these deviations, even if the proper wavenumber dependence of the relaxation times is taken into account.

From the experimental point of view  $p_r$  and  $p_i$  can be obtained from measurements of the phase shift and attenuation of the thermal wave. In order to get an idea about the practicability of measuring the first order deviation in these quantities we take as an example the case of LiF and consider one of the specimens of Berman and Brock (1965) for which the dislocation density  $\delta = 5000$  and the isotope concentration  $\alpha = 0.42$ . For both models the deviations from the classical result at  $T = 20$  K and  $\omega = 10^6$  (frequency  $\sim 160$  kHz) are found to be of the order of 6%

and 7% in  $p_r$  and  $p_i$  respectively. It should certainly be possible to experimentally measure these deviations with a reasonable accuracy.

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