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REDUCTION FORMULAS FOR APPELL FUNCTIONS FOR SPECIAL VALUES OF THE VARIABLES

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Under restrictions on the parameters, expressions for the Appell functions F_1, F_2, and F_3 at $(x, -1)$ and at $(x, \frac{1}{2})$ are obtained. The Mellin-Barnes integral representations and transformation formulas are used in the derivations.

1. INTRODUCTION

A number of cases in which the Appell functions have been shown to reduce to simpler functions for special values of the variables have appeared in the literature; see, for example, Appell and Kampé de Fériet (1926), Bailey (1953), Erdélyi *et al.* (1953), Exton (1976), Gupta (1965), Karlsson (1976), Srivastava (1973). Other general relations which, by specializations, can lead to such formulas appearing in Burchnall and Chaundy (1940, 1941), Erdélyi (1940, 1948), Olsson (1964, 1977), Panda (1974), Singal (1974), Srivastava (1981). Some results for which both variables take on special numerical values are given in Karlsson (1972). The formulas would appear to be classifiable into the reducible types $F(x, -x)$, $F(x, x)$, $F(x, 1)$ and "other". A study of the results indicates gaps in our knowledge; some of these gaps appear to be quite difficult to fill. In this paper we add a few entries to the list of reducibility formulas.

We obtained (see Buschman 1977) identities for the H-functions, among which was one which connected an $H[xy, y]$ to an $H[x, y]$. Since the Appell functions have a Mellin-Barnes integral representation and since the choice of $x = \pm 1$ or $y = 1$, or other choices, lead to the various reducibility classes for the Appell functions it appears that there are connections among these classes and that the integral representations should lead to some results. It should be noted that the H-function identity does not lead to an identity among Appell functions, since the H-functions involved cannot simultaneously be Appell functions. However, further formal manipulations do lead to those formulas for F_1 which were given by Appell and Kampé de Fériet (1926). Rather than use this H-function identity we work directly with the specific simple double contour integrals. These provide an analytic continuation outside of the region of convergence of the defining series. In particular, we obtain some results involving $F(x, -1)$ and consequently for $F(x, \frac{1}{2})$ by use of transformation formulas. Although we obtained (see

Buschman 1979) the H-function identity for functions of more than two variables and this suggests possible generalizations, we here confine ourselves to Appell functions.

The abbreviated notation for quotients of Γ-functions will be used; for example, $\Gamma[\alpha, \beta; \gamma]$ for $\Gamma(\alpha)\Gamma(\beta)/\Gamma(\gamma)$. Since all integrals are of the form $\int_{-i\infty}^{+i\infty}$ taken along lines, with perhaps indentations, we suppress the limits. Notation for the H-function corresponds to that used in Mathai and Saxena (1978).

2. Results at $(-x, -1)$

Expressions can be obtained for F_1 and F_2 which we list. Since the derivations are similar, we outline only the one for F_1.

$$F_1(\alpha, \beta, \beta', 1 + \alpha - \beta'; -x, -1)$$
$$= \tfrac{1}{2}\,\Gamma[1 + \alpha - \beta'; \alpha, \beta]\, H^{12}_{22}\left[x \,\middle|\, \begin{matrix} (1-\beta, 1), (1 - \tfrac{1}{2}\alpha, \tfrac{1}{2}); \\ (0, 1); (\beta' - \tfrac{1}{2}\alpha, \tfrac{1}{2}) \end{matrix}\right]. \qquad ...(2.1)$$

$$F_2(\alpha, \beta, \beta', \gamma, \beta'; -x, -1) = 2^{-\alpha}{}_2F_1(\alpha, \beta, \gamma; -\tfrac{1}{2}x). \qquad ...(2.2)$$

Consider the double contour integral representation for $F_1(-x, -y)$, which converges absolutely for $|\arg(x)| < \pi$, $|\arg(y)| < \pi$ [see Buschman (1977) for convergence tests, for example], and replace y by 1, since this integral representation provides an analytic continuation outside of the convergence region for the series. We use the linear transformation $u = s$, $v = s + t$, which maps the (Im (s), Im (t))-plane onto the (Im (u), Im (v))-plane and for which the Jacobian equals 1. This gives us the representation

$$F_1(\alpha, \beta, \beta', \gamma; -x, -1)$$
$$= \Gamma[\gamma; \alpha, \beta, \beta'] \frac{1}{(2\pi i)^2} \iint \frac{\Gamma(\alpha+v)\Gamma(-u)\Gamma(\beta+u)\Gamma(u-v)\Gamma(\beta'+v-u)}{\Gamma(\gamma + v)} x^u\, du\, dv \qquad ...(2.3)$$

which converges absolutely for $|\arg(x)| < \pi/2$. The integral with respect to v can be evaluated in terms of the Gauss hypergeometric function at -1 to which Kummer's Theorem can be applied, provided $1 + \alpha = \beta' + \gamma$. Proceeding under this additional restriction we have the single integral representation

$$F_1(\alpha, \beta, \beta', 1 + \alpha - \beta'; -x, -1)$$
$$= \tfrac{1}{2}\,\Gamma[1 + \alpha - \beta'; \alpha, \beta] \frac{1}{2\pi i} \int \frac{\Gamma(-u)\,\Gamma(\alpha/2+u/2)\,\Gamma(\beta+u)}{\Gamma(1 + \tfrac{1}{2}\alpha - \beta' + \tfrac{1}{2}u)} x^u\, du \qquad ...(2.4)$$

which gives us (2.1). Since this last integral converges for $|\arg(x)| < \pi$, an analytic continuation is provided by (2.1), and similarly by (2.2) into the cut plane. [Convergence tests are developed in Dixon and Ferrar (1936); preservation of identities under analytic continuation is discussed in Hille (1962).] It should be noted that the duplication formula for the Γ-function allows the result to be expressed in terms of a G_{33}^{23} function, or ultimately as a sum of two ${}_3F_2$ functions.

A difficulty occurs in the evaluation of the single integral in the derivation of the result for F_2, since Kummer's Theorem cannot be used. However, the ${}_2F_1$ is reducible in the degenerate case $\beta' = \gamma'$ (or $\beta = \gamma$). A similar difficulty prevents the method from yielding any such results for F_3. For F_4 there is a problem of convergence of the original double integral and hence with analytic continuation.

3. Other Results

Although reduction formulas involving $F(-x, -x)$ can be obtained by this method, including Appell's formula for F_1, it seems that nothing really new can readily be obtained beyond those results listed in Karlsson (1976), or those which result from those formulas in Burchnall and Chaundy (1940) or those from Erdélyi *et al.* (1953, p. 238). A similar situation occurs for $F(x, -x)$. The difficulties in improving those results seem to be a lack of knowledge of the values of the G-function at ± 1, although some such values are available in Erdélyi *et al.* (1954). A similar problem occurs for $F(-x, -x^2)$.

From the transformation formulas 5.11 (1), (7), (11) of Erdélyi *et al.* (1953) we can next obtain results at $(x, \frac{1}{2})$ from (2.1) and (2.2).

$$\begin{aligned} &F_1(1-\beta', \beta, \beta', 1+\alpha-\beta'; -x, \tfrac{1}{2}) \\ &\quad = (1-x)^{-\beta}\, 2^{\beta'-1}\Gamma[1+\alpha-\beta'; \alpha, \beta] \\ &\qquad \times H_{22}^{12}\left[\frac{x}{1-x} \,\middle|\, \begin{matrix} (1-\beta, 1), (1-\tfrac{1}{2}\alpha, \tfrac{1}{2}); \\ (0, 1); (\beta'-\tfrac{1}{2}\alpha, \tfrac{1}{2}) \end{matrix} \right]. \end{aligned} \qquad \ldots(3.1)$$

$$\begin{aligned} &F_3(\alpha, 1-\beta', \beta, \beta', 1+\alpha-\beta'; -x, \tfrac{1}{2}) \\ &\quad = 2^{\beta'-1}\Gamma[1+\alpha-\beta'; \alpha, \beta] \\ &\qquad \times H_{22}^{12}\left[x \,\middle|\, \begin{matrix} (1-\beta, 1), (1-\tfrac{1}{2}\alpha, \tfrac{1}{2}); \\ (0, 1); (\beta'-\tfrac{1}{2}\alpha, \tfrac{1}{2}) \end{matrix} \right]. \end{aligned} \qquad \ldots(3.2)$$

Similarly, now that $F_1(-x, 1/2)$ is available, we have

$$\begin{aligned} &F_3(1-\beta', \alpha, \beta, \beta', 1+\alpha-\beta'; -x, -1) \\ &\quad = \tfrac{1}{2}(1-x)^{-\beta}\,\Gamma[1+\alpha-\beta; \alpha, \beta] \\ &\qquad \times H_{22}^{12}\left[x \,\middle|\, \begin{matrix} (1-\beta, 1), (1-\tfrac{1}{2}\alpha, \tfrac{1}{2}); \\ (0, 1); (\beta'-\tfrac{1}{2}\alpha, \tfrac{1}{2}) \end{matrix} \right]. \end{aligned} \qquad \ldots(3.3)$$

Since the $F_2(-x, -y)$ would degenerate to a ${}_2F_1$ under the needed restriction on the parameters, that result is omitted.

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References

Appell, P., and Kampé de Fériet, J. (1926). Fonctions hypergéometriques et hypersphériques, Polynomes d'Hermite. Gauthier-Villars, Paris.

Bailey, W. N. (1953). On the sum of a terminating ${}_3F_2(1)$. *Quart J. Math. Oxford* (2), **4**, 237–40.

Burchnall, J. L. (1942). Differential equations associated with hypergeometric functions. *Quart. J. Math. Oxford* (1), **13**, 90–106.

Burchnall, J. L., and Chaundy, T. W. (1940). Expansions of Appell's double hypergeometric functions, I. *Quart. J. Math. Oxford* (1), **11**, 249–70.

——— (1941). Expansions of Appell's double hypergeometric functions, II. *Quart. J. Math. Oxford* (1), **12**, 112–28.

Buschman, R. G. (1977). H-functions of two variables, II. *Jñānabha.* (Prof. Charles Fox Memorial Volume), **7**, 105–16.

——— (1979). H-functions of N variables. *Ranchi Univ. Math. J.*, **10**, to appear.

Dixon, A. L., and Ferrar, W. L. (1936). A class of discontinuous integrals. *Quart. J. Math. Oxford* (1), **7**, 81–96.

Erdélyi, A. (1940). Some confluent hypergeometric functions of two variables. *Proc. R. Soc. Edinb.*, **60**, 344–61.

——— (1948). Transformations of hypergeometric functions of two variables. *Proc. R. Soc. Edinb.*, A**62**, 378–85.

Erdélyi, A., Magnus, W., Oberhettinger, F., and Tricomi, F. G. (1953). Higher Transcendental Functions, Vol. I. McGraw-Hill Book Co., Inc., New York.

——— (1954). Tables of Integral Transforms, Vols. 1, 2. McGraw-Hill Book Co., Inc., New York.

Exton, Harold (1976). Multiple Hypergeometric Functions and Applications. Ellis Horwood Ltd., Chichester.

Gupta, K. C. (1965). A reduction formula for Appell's function F_4. *Ganita*, **16**, 81–82.

Hille, E. (1962). Analytic Function Theory, Vol. 2. Ginn.

Karlsson, Per W. (1972). Some reduction formulas for hypergeometric functions. *Math. Scand.*, **31**, 248–56.

——— (1976). On Certain Generalizations of Hypergeometric Functions. Physics Laboratory II, Technical University of Denmark, Lyngby.

Mathai, A. M., and Saxena, R. K. (1978). The H-function with Applications in Statistics and other Disciplines. Wiley Eastern Ltd., New Delhi.

Olsson, P. O. M. (1964). Integration of the partial differential equations for the hypergeometric functions F_1 and F_D of two or more variables. *J. Math. Phys.*, **5**, 420–30.

——— (1977). On the integration of the differential equations of five parameter double-hypergeometric functions of second order. *J. Math. Phys.*, **18**, 1285–93.

Panda, Rekha (1974). Some multiple series tansformations. *Jñānabha*, **A4**, 165–68.

Slater, L. J. (1966). Generalized Hypergeometric Functions. Cambridge University Press, Cambridge.

Singal, B. M. (1974). On the reducibility of Lauricella's function F_D. *Jñānabha*, **A4**, 163–64.

Srivastava, H. M. (1973). On the reducibility of Appell's function F_4. *Canad. Math. Bull.*, **16**, 295–98.

——— (1981). Some generalizations of Carlson's identity. *Boll. Un. Mat. Ital.*, (5) **18A**, 138–43.