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SPHERICALLY SYMMETRIC SPACE-TIMES IN BIMETRIC RELATIVITY THEORY—II

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The stiff matter or the radiation field are the only possible matter fields agreeable to bimetric relativity for a spherically symmetric isotropic fluid distribution having an equation of state that pressure and energy density are in constant ratio. Furthermore the nonexistence of dust distribution is established in a space-time where the unique choice of time is made.

1. INTRODUCTION

A new theory of gravitation has been proposed by Rosen (1973, 1974, 1977) which is known as bimetric theory of gravitation or bimetric relativity (BR) theory. At each point of space-time, BR defines two metric tensors: a Riemannian metric tensor g_{ij} and the background flat space-time metric tensor f_{ij}. The tensor g_{ij} describes the geometry of a curved space-time and thus the gravitational fields. The background metric f_{ij} refers to inertial forces. [However, the recent work of Rosen (1978) differs from his earlier above mentioned views about f_{ij}. He feels that f_{ij}, instead taking it as a background flat space-time metric, should be considered from the cosmological considerations.] The postulation of f_{ij} improves the formalism of general relativity (GR) without changing its physical content. Certain non-tensorial quantities can be attributed tensorial properties, an example of which is the pseudo energy tensor of the gravitational field. Also BR satisfies the covariance and equivalence principles : the foundations of GR. The theory agrees with present observational facts to the degree as GR. However, it does not predict black holes and thus avoids the breakdown of the physical concept of space-time involved. As in GR, the variational principle also leads to the conservation law

$$T^{ij}_{;j} = 0 \qquad \ldots(1.1)$$

where (;) denotes covariant differentiation with respect to g_{ij}. Accordingly the geodesic equation of a test particle is the same as that of GR. The field equations of BR, derived from variational principle, lead to

$$K^j_i = N^j_i - \tfrac{1}{2}\,\delta^j_i\,N = -8\,\pi k T^j_i \qquad \ldots(1.2)$$

$$N^j_i = \tfrac{1}{2} f^{ab}\,(g^{jh}\,g_{ih}\mid_a)\mid_b$$

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$k = (g/f)^{1/2}$, $g = \det(g_{ij})$ and $f = \det(f_{ij})$.

A bar (|) stands for the f-covariant differentiation. In case f_{ij} is taken as a Lorentz metric : diag $(-1, -1, -1, 1)$, f-differentiation reduces to the usual partial differentiation.

Though the proposed theory BR and GR at present are observationally indistinguishable, there are instances of some sharp differences between them. Some of them are already mentioned in the above. The present author has noted these differences in his work* (see Karade 1980). This report is the continuation of our work (Karade 1980) on spherically symmetric (SS) space-times of BR. Here two interesting results are established for the distribution of perfect fluid obeying certain equation of state. In particular the non-existence of SS dust distribution in BR is obtained which contrasts sharply with GR.

2. The Field Equations

The most general SS line-element can be reduced to

$$ds^2 = -e^A dr^2 - e^B d\Sigma^2 + e^C dt^2$$

$$d\Sigma^2 = d\theta^2 + \sin^2\theta\, d\phi^2 \qquad \text{...(2.1)}$$

and A, B, C are functions of r and t.

The flat space-time corresponding to (2.1) is

$$d\sigma^2 = -dr^2 - r^2 d\Sigma^2 + dt^2. \qquad \text{...(2.2)}$$

The non-vanishing components of K_i^j etc. are detailed in Karade (1980). Consider a fluid distribution specified by pressure p and density ϵ. In the comoving reference system, the components of energy momentum tensor are

$$T_1^1 = T_2^2 = T_3^3 = -p \text{ and } T_4^4 = \epsilon. \qquad \text{...(2.3)}$$

By the substitution $e^B = r^2 e^A$, the line-element (2.1) is transformed to an isotropic form

$$ds^2 = -e^A(dr^2 + r^2 d\Sigma^2) + e^C dt^2. \qquad \text{...(2.4)}$$

This form (2.4) of the line-element and (2.3) for matter tensor are used by Rosen (1977) to obtain a stellar model. Now the components of K_i^j get simplified and yield

$$\left.\begin{aligned} K_1^1 &= K_2^2 = K_3^3 = (1/4r)\,[(r\alpha)'' - (r\alpha)^{\cdot\cdot}] \\ K_4^4 &= (1/4r)\,[(r\beta)'' - (r\beta)^{\cdot\cdot}] \end{aligned}\right\} \qquad \text{...(2.5)}$$

*This includes unpublished work also.

where $\alpha = A + C$, $\beta = 3A - C$.

The prime ($'$) and the dot ($\cdot$) denote the partial derivative with respect to r and t respectively.

Now the field equations (1.2) and the conservation equation (1.1) give

$$(r\alpha)'' - (r\alpha)^{\cdot\cdot} = 32\pi r p e^{(3\alpha+\beta)/4} \quad ...(2.6)$$

$$(r\beta)'' - (r\beta)^{\cdot\cdot} = -32\pi r \epsilon e^{(3\alpha+\beta)/4} \quad ...(2.7)$$

$$3\dot{A}\,(p + \epsilon) + 2\dot{\epsilon} = 0, \quad C'(p + \epsilon) + 2p' = 0. \quad ...(2.8)$$

Supposing that p is a known function of ϵ, (2.8) gives (see Landau and Lifshitz 1971)

$$3A = -2 \int \frac{d\epsilon}{p + \epsilon} + R(r) \quad ...(2.9)$$

and

$$C = -2 \int \frac{dp}{p + \epsilon} + T(t) \quad ...(2.10)$$

where the functions R and T are arbitrary functions of r and t respectively.

Let us suppose that the fluid distribution satisfies the equation of state

$$-p/\epsilon = \text{constant } (= n), \text{ say.}$$

From (2.6) and (2.7), we arrive at

$$\frac{(r\alpha)'' - (r\alpha)^{\cdot\cdot}}{(r\beta)'' - (r\beta)^{\cdot\cdot}} = n$$

which after integration yields

$$r(\alpha - n\beta) = f(r + t) + g(r - t) \quad ...(2.11)$$

f and g being functions of their arguments.

Equation (2.11) then gives

$$(1 - 3n)\,A + (1 + n)\,C = (f + g)/r. \quad ...(2.12)$$

Also (2.9) and (2.10) imply

$$3nA + C = nR + T. \quad ...(2.13)$$

Solving (2.12) and (2.13), we get

$$A = X\,[(1 + n)\,(nR + T) - (f + g)/r] \quad ...(2.14)$$

$$C = nX\,[(3n + 7)\,(nR + T) - 3\,(f + g)/r] \quad ...(2.15)$$

$$X = (3n^2 + 6n - 1)^{-1}, \quad n \neq -1 \pm 2(3)^{-1/2}.$$

The pressure and density turn out to be

$$32\pi p = UYe^{-Z} \qquad \text{...(2.16)}$$

$$32\pi\epsilon = -UVe^{-Z} \qquad \text{...(2.17)}$$

where $\quad U = n(R'' + 2R'/r) - \ddot{T}$

$$V = -X(3n^2 + 4n - 3),\ Y = X(3n^2 + 8n + 1)$$

and $\quad Z = \frac{1}{2} X\,[(3n^2 + 10n + 3)\,(nR + T) - 3(1 + n)\,(f + g)/r].$

But $-p/\epsilon = n$ and hence (2.16) and (2.17) yield

$$(n + 1)^2\,(3n + 1) = 0 \text{ i.e., } n = -1 \text{ or } -1/3.$$

Therefore the proposed equation of state specifies the nature of fluid by

$$p = \epsilon \qquad \text{...(2.18)}$$

or

$$p = \epsilon/3. \qquad \text{...(2.19)}$$

Equation (2.18) stands for a 'stiff' matter distribution [fluid satisfying (2.18) is also termed as Zeldovich fluid] whereas (2.19) refers to radiation. There are no other alternatives except but stiff matter or radiation distribution. An important result thus follows:

A SS isotropic fluid distribution, in a comoving frame, where pressure and energy density are in a constant ratio gives either stiff matter or radiation continuum.

For stiff matter, we obtain

$$A = (f + g)/4r$$

$$C = T - R - 3A$$

$$32\pi p = 32\pi\epsilon = -\,(R'' + \ddot{T} + 2R'/r)\,e^{(R-T)/2}.$$

On physical grounds, the energy density must be positive and hence the functions R and T should be chosen so as to satisfy

$$R'' + \ddot{T} + 2R'/r < 0.$$

For radiation, we get

$$A = \frac{R - 3T}{12} + \frac{3(f + g)}{8r},$$

$$C = -\,\frac{R - 3T}{4} - \frac{3(f + g)}{8r},$$

$$32\pi p = (32\pi\epsilon)/3 = -\tfrac{1}{6}(R'' + 3\ddot{T} + 2R'/r)e^{-3(f+g)/8r},$$

$$R'' + 3\ddot{T} + 2R'/r < 0.$$

3. Dust Distribution

For dust $p = 0$ and then (2.8) gives

$$C' = 0. \qquad \ldots(3.1)$$

Following Landau and Lifshitz (1971), for a unique choice of the time t, we can set

$$C = 0. \qquad \ldots(3.2)$$

Then the components of K_i^j bear the relation

$$K_1^1 + 2K_2^2 = K_4^4$$

and hence

$$T_1^1 + 2T_2^2 = T_4^4 \qquad \ldots(3.3)$$

yielding $\quad \epsilon = 0. \qquad \ldots(3.4)$

Therefore an isotropic distribution of dust is not admitted in BR where the space-time is separated into a space and a universal time orthogonal thereto.

It is interesting that Rosen (1977) has obtained an isotropic dust solution with $A = A(t)$ and $C = C(t)$ concluding no singularity because the collapse goes over smoothly into the expansion. Thus one may have an impression that our findings contradict Rosen's (1977). But this is not true. The root cause of an apparent contradiction lies in the very concept of time adopted in the two procedures. Our concept of unique time, defined by (3.2), differs from that of Rosen (1977). In our case the coordinate time identifies with proper time measured by any local observer, with his local clock, at rest with respect to the mean motion of matter in his neighbourhood and thus the astronomical time is appropriately expressed in terms of the coordinate time t. [For details one may refer to Tolman (1934).] It is easy to verify that if our concept of universal time is applied to Rosen's formalism (for Rosen this amounts to $\phi = 0$), the result (3.4) emerges and the contradiction disappears.

As a matter of fact, the solution of (3.1) is $C = C(t)$.

Defining a new time coordinate $\bar{t}$ as

$$\bar{t} = \int e^{C/2}\, dt \qquad \ldots(3.5)$$

the line-element (2.4) takes the form

$$ds^2 = -A(r, \bar{t})\,(dr^2 + r^2 d\,\Sigma^2) + d\bar{t}^2. \qquad \ldots(3.6)$$

[In fact Rosen (1977) has employed (3.5).]

Also with (3.2), the line-element (2.4) assumes the form

$$ds^2 = -A(r, t)\,(dr^2 + r^2 d\,\Sigma^2) + dt^2. \qquad \ldots(3.7)$$

In this section the vanishing of the energy density has been established in BR w.r.t. (3.7) and not w.r.t. (3.6). Though all the coordinate systems are admissible and the forms (3.6) and (3.7) are similar, one should not haste to conclude $\epsilon = 0$ for (3.6). The reason is that with (3.6), the background metric is

$$ds^2 = -dr^2 - r^2 d\,\Sigma^2 + d\bar{t}^2 \qquad \ldots(3.8)$$

and not (2.2). When the transformation (3.5) is applied, the background metric (2.2) gets changed. (This is agreeable to Rosen*.) The contradictory results are obtained when for a curved space-time we consider the metric (3.6) with time coordinate as $\bar{t}$ and in the background we, without modification owing to (3.5), adopt the same metric (2.2) with time t. Thus in one formalism we adopt two different times which is a serious matter. In GR this never arises because of a single metric formulation.

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*Private communication with Prof. Rosen on similar matter in other context.