*Indian J. pure appl. Math.*, 12(7) : 799–803, July 1981

# ON DATTA'S BITOPOLOGICAL PARACOMPACTNESS

CH. KONSTADILAKI-SAVOPOULOU
*Department of Mathematics, Aristotle University of Thessaloniki, Greece*

AND

I. L. REILLY
*Department of Mathematics, University of Auckland, New Zealand*

(*Received* 23 *July* 1980)

In this paper we examine recent definitions of paracompactness for bitopological spaces given by Datta. We show that his attempts to obtain a definition which does not collapse a pairwise Hausdorff bitopological space into a topological space are only marginally more successful than previous ones. As a consequence, the application of his ideas to an outstanding problem on quasi-metric spaces is invalid.

## 1. INTRODUCTION

Since the formal study of bitopological spaces began with the paper of Kelly (1963) considerable effort has been expended in obtaining appropriate generalizations of standard topological properties in the bitopological category. The covering properties have proved to be the most difficult in this respect. Some of the problems of, and alternative definitions for, bitopological compactness have been discussed by Cooke and Reilly (1975). A similar paper dealing with bitopological local compactness is that of Mrsevic (1980).

Paracompactness seems to have been the most intractable of all. The major difficulty with bitopological paracompactness was indicated in the early paper of Fletcher *et al.* (1969). It is to obtain a definition of pairwise paracompactness which does not, in the presence of the pairwise Hausdorff property, collapse the bitopological space to a topological space by requiring the two topologies to coincide. In a more recent paper, Datta (1977) claimed to have avoided this pitfall, but does not bother to give an example of a non-trivial bitopological space $(X, \tau_1, \tau_2)$ satisfying his definition(s), and for which $\tau_1 \neq \tau_2$. In this paper, we examine Datta's definitions and show that it will not be easy to produce such examples. Indeed, we show that Datta's strongly pairwise paracompact property suffers the fate that he is trying to avoid, and that for his pairwise paracompact spaces the topologies coincide on important subspaces. This means that several of his subsequent theorems are not bitopological results at all, but known topological ones. In particular, his partial solution to a question of Stoltenberg (1969), namely "When is a quasi-metric topological space paracompact?", is no answer at all.

## 2. Pairwise Paracompact Spaces

We first give the necessary definitions.

*Definition* 1 (Fletcher *et al.* 1969) — A cover $\mathcal{U}$ of a bitopological space $(X, \tau_1, \tau_2)$ is pairwise open if $\mathcal{U} \subset \tau_1 \cup \tau_2$, and $\tau_1 \cap \mathcal{U}$ contains a non-empty set and $\tau_2 \cap \mathcal{U}$ contains a non-empty set.

The next four definitions are those of Datta (1977, Definitions 2.1, 2.2, 2.4 and 2.5). Definition 2 was given by Swart (1971, Definition 4.1), who called such a cover a $\tau_1\tau_2$ open cover.

*Definition* 2 — A cover $\mathcal{U}$ of $(X, \tau_1, \tau_2)$ is weakly pairwise open if it consists of either $\tau_1$ open sets or $\tau_2$ open sets or both.

*Definition* 3 — If $\mathcal{U}$ is a (weakly) pairwise open cover of $(X, \tau_1, \tau_2)$, then the (weakly) pairwise open cover $\mathcal{V}$ of $(X, \tau_1, \tau_2)$ is a parallel refinement of $\mathcal{U}$ if every $\tau_1$ open set in $\mathcal{V}$ is contained in some $\tau_1$ open set of $\mathcal{U}$ and every $\tau_2$ open set in $\mathcal{V}$ is contained in some $\tau_2$ open set of $\mathcal{U}$.

*Definition* 4 — A (weakly) pairwise open cover $\mathcal{U}$ of $(X, \tau_1, \tau_2)$ is pairwise locally finite if each point of $X$ has a $\tau_1$ open ($\tau_2$ open) neighbourhood which meets only finitely many of the $\tau_2$ open ($\tau_1$ open) sets of $\mathcal{U}$.

*Definition* 5 — A refinement $\mathcal{V}$ of a (weakly) pairwise open cover $\mathcal{U}$ of $(X, \tau_1, \tau_2)$ is pairwise locally finite if each point of $X$ has a $\tau_1$ open ($\tau_2$ open) neighbourhood which meets only finitely many of the sets of $\mathcal{V}$, and these form a refinement of the family of $\tau_2$ open ($\tau_1$ open) sets of $\mathcal{U}$.

We differ from Fletcher *et al.* (1969) and Datta (1977) in that we do not require a bitopological space to be pairwise Hausdorff in order to be pairwise paracompact. We do this firstly for philosophical reasons, and secondly because it clarifies our subsequent results.

*Definition* 6 — A bitopological space $(X, \tau_1, \tau_2)$ is FHP-pairwise paracompact if every $\tau_1$ open cover of $X$ has a $\tau_2$ open $\tau_2$ locally finite refinement and every $\tau_2$ open cover of $X$ has a $\tau_1$ open $\tau_1$ locally finite refinement.

*Definition* 7 — A bitopological space $(X, \tau_1, \tau_2)$ is (strongly) $D$-pairwise paracompact if for each (weakly) pairwise open cover of $X$ there is a pairwise locally finite (weakly) pairwise open parallel refinement.

*Definition* 8 [Raghavan and Reilly (1977, Definition 2)] — In the bitopological space $(X, \tau_1, \tau_2)$, we say that $\tau_1$ is paracompact with respect to $\tau_2$ if each $\tau_1$ open cover of $X$ has a $\tau_1$ open refinement which is $\tau_2$ locally finite.

It was shown by Fletcher *et al.* (1969, Theorem 9) that FHP-pairwise paracompactness was inadequate in the following sense.

*Theorem* 1 — If $(X, \tau_1, \tau_2)$ is pairwise Hausdorff and FHP-pairwise paracompact then $\tau_1 = \tau_2$.

This is the starting point of Datta's paper (1977), which is an attempt to provide a notion of bitopological paracompactness without this failing. The results we now prove show that he has been successful to a limited extent only. The next two results are an elaboration of Proposition 2.7 of Datta (1977). In our proofs, whenever necessary we adopt the convention of Datta (1977, p. 686). This means that a set which is both $\tau_1$ and $\tau_2$ open is considered throughout as either $\tau_1$ open or $\tau_2$ open. Its openness in the other topology is disregarded in the subsequent discussion.

*Proposition* 1 — If $(X, \tau_1, \tau_2)$ is strongly $D$-pairwise paracompact then $\tau_1$ is paracompact with respect to $\tau_2$.

PROOF : Let $\mathcal{U}$ be any $\tau_1$ open cover of $X$. If any member of $\mathcal{U}$ is also $\tau_2$ open, that fact is disregarded according to the convention outlined above. So $\mathcal{U}$ is a weakly pairwise open cover of $X$, and hence there is a pairwise locally finite weakly pairwise open parallel refinement $\mathcal{V}$ of $\mathcal{U}$. By the convention, there are no non-empty $\tau_2$ open sets in $\mathcal{U}$, and hence there are no non-empty $\tau_2$ open sets in $\mathcal{V}$ since it is a parallel refinement of $\mathcal{U}$. Thus $\mathcal{V}$ is a family of $\tau_1$ open sets, and since it is pairwise locally finite, each point in $X$ has a $\tau_2$ open neighbourhood which meets only finitely many of the members of $\mathcal{V}$. Thus $\tau_1$ is paracompact with respect to $\tau_2$.

*Proposition* 2 — If $(X, \tau_1, \tau_2)$ is $D$-pairwise paracompact and $F$ is a proper $\tau_2$ closed subset of $X$, then $\tau_1 \mid F$ is paracompact with respect to $\tau_2 \mid F$ in the bitopological subspace $(F, \tau_1 \mid F, \tau_2 \mid F)$.

PROOF : Let $\mathcal{U}$ be any $\tau_1 \mid F$ open cover of $F$. Then $\mathcal{U} = \{U \cap F : U$ is in some family $\mathcal{U}^*$ of $\tau_1$ open sets$\}$. Then $\mathcal{U}^* \cup \{X - F\}$ is a pairwise open cover of $X$, and so it has a pairwise locally finite pairwise open parallel refinement $\mathcal{V}^*$. By the convention, every non-empty $\tau_2$ open set in $\mathcal{V}^*$ is contained in $X - F$. Since $\mathcal{V}^*$ covers $X$, the $\tau_1$ open members of $\mathcal{V}^*$ cover $F$. If $\mathcal{V} = \{V \cap F : V \in \mathcal{V}^* \cap \tau_1\}$, then $\mathcal{V}$ is a $\tau_1 \mid F$ open refinement of $\mathcal{U}$ which is $\tau_2 \mid F$ locally finite, since $\mathcal{V}^*$ is pairwise locally finite. Hence $\tau_1 \mid F$ is paracompact with respect to $\tau_2 \mid F$.

By interchanging the roles of $\tau_1$ and $\tau_2$, we obtain a similar result for a proper $\tau_1$ closed subset of $X$.

The Definition 8 suggests a natural way to define bitopological paracompactness different from the definition of Fletcher *et al.* (1969) and those of Datta (1977). Namely, $(X, \tau_1, \tau_2)$ is pairwise paracompact if $\tau_1$ is paracompact with respect to $\tau_2$ and $\tau_2$ is paracompact with respect to $\tau_1$. Propositions 1 and 2 imply that this definition is essentially weaker than Datta's definitions. In obtaining results on the comparison of topologies for quite another purpose, Raghavan and Reilly (1977, Proposition 4) proved a result which is crucial to this discussion.

*Proposition* 3 — If $(X, \tau_1, \tau_2)$ is pairwise Hausdorff and $\tau_1$ is paracompact with respect to $\tau_2$, then $\tau_1 \subset \tau_2$.

These three Propositions have far-reaching effects on Datta's definitions and his results.

*Theorem* 2 — If $(X, \tau_1, \tau_2)$ is pairwise Hausdorff and strongly $D$-pairwise paracompact, then $\tau_1 = \tau_2$.

PROOF : It follows immediately from Propositions 1 and 3.

*Theorem* 3 — If $(X, \tau_1, \tau_2)$ is pairwise Hausdorff and $D$-pairwise paracompact, and if $F$ is a proper $\tau_2$ closed subset of $X$, then $\tau_1 \mid F \subset \tau_2 \mid F$ in the bitopological subspace $(F, \tau_1 \mid F, \tau_2 \mid F)$.

PROOF : It follows immediately from Propositions 2 and 3, and the fact that the pairwise Hausdorff property is hereditary.

*Corollary* 1 — If $(X, \tau_1, \tau_2)$ is pairwise Hausdorff and $D$-pairwise paracompact, and if $F$ is a proper subset of $X$ which is $\tau_1$ closed and $\tau_2$ closed, then $\tau_1 \mid F = \tau_2 \mid F$ in the bitopological subspace $(F, \tau_1 \mid F, \tau_2 \mid F)$.

Our Theorem 2 provides an immediate proof of more than half of Datta's Theorem 2.10, namely his sequence of implications (a′) implies (b′) implies (c′) implies (d), and without a pairwise regular hypothesis on $(X, \tau_1, \tau_2)$ or Hausdorff conditions on $(X, \tau_1)$ and $(X, \tau_2)$. Unfortunately, it destroys the significance of these results by requiring that $\tau_1 = \tau_2$. Datta uses the result that (a′) implies (d) to obtain a partial answer to a question raised by Stoltenberg (1969) about conditions which are sufficient for a quasi-metric topological space to be paracompact. In view of our Theorem 2, Datta's Theorem 2.11 states that a quasi-metric topology is paracompact if it is Hausdorff and paracompact and coincides with its conjugate topology. Indeed, almost all the material on the final two pages of Datta's (1977) paper is rendered pointless by the result of Theorem 2 above.

Since we do not demand pairwise Hausdorff as part of the definition of pairwise paracompactness, we now consider to what extent Datta's Theorem 2.10 can be salvaged for pairwise regular spaces. The following result, which is similar to our Proposition 3 above, follows immediately from Lemma 1 and Proposition 2 of Raghavan and Reilly (1978). We recall that a topological space $(X, \tau)$ is $R_0$ if it satisfies one of the following equivalent conditions:

(i) $x \in U \in \tau$ implies $\text{cl}\,\{x\} \subset U$,

(ii) $\text{cl}\,\{x\} = \cap\,\{U : U \text{ is an open neighbourhood of } x\}$ for each $x \in X$,

(iii) $\text{cl}\,\{x\} = \text{cl}\,\{y\}$ or $\text{cl}\,\{x\} \cap \text{cl}\,\{y\} = \phi$ for $x, y \in X$.

*Theorem* 4 — If $(X\ \tau_1, \tau_2)$ is pairwise regular and strongly $D$-pairwise paracompact, and $(X, \tau_1)$ and $(X, \tau_2)$ are $R_0$ spaces then $\tau_1 = \tau_2$.

*Theorem* 5 — If $(X, \tau_1, \tau_2)$ is $D$-pairwise paracompact, $\tau_1$ is regular with respect to $\tau_2$, $(X, \tau_1)$ is $R_0$, and if $F$ is a proper $\tau_2$ closed subset of $X$, then $\tau_1 \mid F \subset \tau_2 \mid F$ in the bitopological subspace $(F, \tau_1 \mid F, \tau_2 \mid F)$.

*Corollary* 2 — If $(X, \tau_1, \tau_2)$ is $D$-pairwise paracompact and pairwise regular, $(X, \tau_1)$ and $(X, \tau_2)$ are $R_0$ spaces, and if $F$ is a proper subset of $X$ which is $\tau_1$ closed and $\tau_2$ closed, then $\tau_1 \mid F = \tau_2 \mid F$ in the bitopological subspace $(F, \tau_1 \mid F, \tau_2 \mid F)$.

These results show that Datta's Theorem 2.10 cannot be saved for pairwise regular spaces in which both of the topologies are $R_0$. This is a much weaker requirement than the Hausdorff conditions of Datta's (c') implies (d).

## Acknowledgement

The second author wishes to acknowledge the hospitality of the Department of Mathematics, University College, London, and the financial support of the Aristotle University of Thessaloniki and the Democritus University of Thrace.

## References

Cooke, I. E., and Reilly, I. L. (1975). On bitopological compactness. *J. Lond. math. Soc.* (2), **9**, 518–22.

Datta, M. C. (1977). Paracompactness in bitopological spaces and an application to quasi-metric spaces. *Indian J. pure appl. Math.*, **8**, 685–90.

Fletcher, P., Hoyle, H. B., and Patty, C. W. (1969). The comparison of topologies. *Duke Math. J.*, **36**, 325–32.

Kelly, J. C. (1963). Bitopological spaces. *Proc. Lond. math. Soc.*, **13**, 71–89.

Mrsevic, M. (1980). On bitopological local compactness. *Mat. Vesnik*, (to appear).

Raghavan, T. G., and Reilly, I. L. (1977). Metrizability of quasi-metric spaces. *J. Lond. math. Soc.* (2), **15**, 169–72.

——— (1978). On non-symmetric topological structures. *Colloq. Math. Soc. Janos Bolyai, Budapest*, **23**, 1005–14.

Stoltenberg, R. A. (1969). On quasi-metric spaces. *Duke Math. J.*, **36**, 65–71.

Swart, J. (1971). Total disconnectedness in bitopological spaces and product bitopological spaces. *Indag. Math.*, **33**, 135–45.