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LAGUERRE FUNCTION OF A HYPERSURFACE IN A FINSLER SPACE

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In this paper, an operator ∇ is introduced to generalise the Laguerre function in a hypersurface of a Finsler space in order to determine an equation for the generalised Laguerre line in the hypersurface.

§1. Nirmala (1964) has introduced an operator ∇ in the hypersurface of a Riemannian space and shown that if **E** is the unit tangent vector of a curve of the hypersurface and **N** its unit normal vector, then

$$-\mathbf{E}\,.\,\frac{\nabla \mathbf{N}}{\nabla s}\,\mathbf{E} = \Omega_{\alpha\beta;\gamma}\, \nu^{\alpha}\nu^{\beta}\nu^{\gamma} \qquad \ldots(1.1)$$

where $\Omega_{\alpha\beta}$ are the components of the second fundamental tensor and ν^{α} are the components of the vector **E** in the hypersurface. It is easy to verify that at an umbilical point of the hypersurface, the expression on the right-hand side of (1.1) becomes dk_n/ds. This shows that if the curve lies on an umbilical hypersurface then the scalar on left-hand side of (1.1) will vanish if and only if the normal curvature of the hypersurface in the direction of this curve is the same at each of its points.

The observations made in the preceding paragraph have motivated Nirmala (1964, 1965) for defining the generalised Laguerre function by replacing the vector **N** in (1.1) by an arbitrary vector field $\underline{\lambda}$ tangent to a congruence of curves in F_n. The curve along which this function vanishes has been called generalised Laguerre line of the hypersurface.

In this paper we shall introduce an operator ∇ which will enable us to define generalised Laguerre function in the hypersurface of a Finsler space. The expression for the Laguerre function in terms of curvature of a congruence in F_n and of a curve in F_{n-1} will be obtained. The purpose of finding expression for the generalised Laguerre function is to determine an equation for the generalised Laguerre line in the hypersurface of a Finsler space. This curve is a generalisation to the Finsler space of the corresponding curve of the Riemannian hypersurface and also of the curve mentioned in the first paragraph of the introduction.

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§2. Let us consider a hypersurface F_n referred to a local coordinate system $x^i (i = 1, 2, 3, \ldots, n)$ and use the notations and definitions given by Rund (1956, 1959). Consider a set of $(m-n)$ congruences of curves each of which passes through each point of the subspace. Since the vectors, in general, are not normal to the surface, the contravariant components of a unit vector in the direction of the curve of the congruence ν^i may be expressed linearly in terms of X^i_α and the set of normals to the subspace. Thus resolving ν tangentially and normally to F_{n-1} we get

$$\nu^i = t^\alpha X^i_\alpha + r n^i. \qquad \ldots(2.1)$$

In case of a hypersurface there exist two systems of normals, n^{*i} depending on the directions x'^i and n^i independent of the direction argument. The covariant derivatives of these two types of normals as given by Rund (1956) are

$$n^i_{;\beta} = -\gamma^{\alpha\delta}\Omega_{\alpha\beta}X^j_\delta - C_{ihk}n^h X^k_\beta \{g^{ij}(x, n) - \tfrac{1}{2} n^i n^j\} \qquad \ldots(2.2)$$

and

$$n^{*j}_{;\beta} = -\psi(x, x')\, g^{\alpha\delta}(u, u')\, \Omega^*_{\alpha\beta}\, X^j_\delta + \frac{n^{*j}(x, x')}{2\psi(x, x')}\psi_{;\beta}$$

$$- C^*_{ihk}\, n^{*h}(x, x')\, X^k_\beta \left\{g^{ij}(x, x') - \frac{n^{*i}(x, x')\, n^{*j}(x, x')}{2\psi(x, x')}\right\} \qquad \ldots(2.3)$$

where $\psi(x, x') = g_{ij}(x, x')\, n^{*i}(x, x')\, n^{*j}(x, x'),\quad E_{ihk} = g_{ih;k}(x, x')$

and $\gamma^{(n)}_{\alpha\beta} = g^{(x,n)}_{ij} X^i_\alpha X^j_\beta,\ \gamma^{\alpha\theta}\gamma_{\alpha\beta} = \delta^\theta_\beta$.

From (2.2) we have

$$\frac{\delta n^i}{\delta s} = -\gamma^{\alpha\delta}\Omega_{\alpha\beta}X^i_\delta \frac{du^\beta}{ds} + \tfrac{1}{2} n^i C_{jhk} n^j n^h \frac{dx^k}{ds} - C_{jhk} n^h \frac{dx^k}{ds} g^{ij}(x, n). \qquad \ldots(2.4)$$

We write (2.4) in the form,

$$\frac{\delta n^i}{\delta s} = A^\alpha X^i_\alpha + D n^i. \qquad \ldots(2.5)$$

Multiplying (2.5) by $g_{ij}(x, n)\, n^j$ we get

$$D = g_{ij}(x, n)\, n^j \frac{\delta n^i}{\delta s} = \tfrac{1}{2} C_{jhk} n^j n^h \frac{dx^k}{ds}. \qquad \ldots(2.6)$$

Again multiplying (2.4) by $g_{ij}(x, n)\, X^j_\beta$ we have

$$g_{ij}(x, n) \frac{\delta n^i}{\delta s} X^j_\beta = g_{ij}(x, n)\, A^\alpha X^i_\alpha X^j_\beta$$

$$= A^\alpha \gamma_{\alpha\beta}$$

or $$g_{ij}(x, n) \frac{\delta n^i}{\delta s} X^j_\beta \gamma^{\beta\theta} = \delta^\theta_\alpha A^\alpha = A^\theta. \qquad \ldots(2.7)$$

Therefore, from (2.4), (2.5) and (2.7) we may have

$$\begin{aligned} A^\varphi &= \gamma^{\beta\varphi} g_{ij}(x, n) \frac{\delta n^i}{\delta s} X^j_\beta \\ &= \gamma^{\beta\varphi} g_{ij}(x, n) X^j_\beta \left[-\gamma^{\alpha\varepsilon} \Omega_{\alpha\theta} X^i_\varepsilon \frac{du^\theta}{ds} - C_{lhk} n^h \frac{dx^k}{ds} g^{il} \right] \\ &= -\left(\gamma^{\beta\varphi} \Omega_{\beta\theta} \frac{du^\theta}{ds} + \gamma^{\beta\varphi} C_{jhk} X^j_\beta n^h \frac{dx^k}{ds} \right). \end{aligned}$$

Thus, $$A^\alpha = -\gamma^{\beta\alpha} \left(\Omega_{\beta\theta} \frac{du^\theta}{ds} + C_{jhk} X^j_\beta n^h \frac{dx^k}{ds} \right). \qquad \ldots(2.8)$$

§3. Let v^α and E^i be the contravariant components of the unit tangent vector to the curve C in F_{n-1} and F_n respectively where $C: x^\alpha = x^\alpha(s)$ is any curve in F_{n-1}. The intrinsic derivative or the derived vector of ν in the direction of C is denoted by $\frac{\delta \nu}{\delta s}$ and is given by

$$\frac{\delta \nu^i}{ds} = \nu^i_{;\beta} \frac{dx^\beta}{ds}. \qquad \ldots(3.1)$$

The tensor derivative of (2.1) after some simplifications can be written as

$$\nu^i_{;\alpha} = t^\beta_{;\alpha} X^i_\beta + t^\beta I^i_{\alpha\beta} + r n^i_{;\alpha} + r_{;\alpha} n^i \qquad \ldots(3.2)$$

where $$I^i_{\alpha\beta} = \sum_\mu A_{(\mu)\alpha\beta}\, n^i_{(\mu)} + \omega^i_{\alpha\beta}$$

as defined by Rund (1956). We write eqn. (3.2) as

$$\nu^i_{;\alpha} v^\alpha = t^\beta_{;\alpha} v^\alpha X^i_\beta + t^\beta I^i_{\alpha\beta} v^\alpha + r n^i_{;\alpha} v^\alpha + r_{;\alpha} v^\alpha n^i. \qquad \ldots(3.3)$$

Using (3.1) eqn. (3.3) can be written as

$$\frac{\delta \nu^i}{\delta s} = \frac{\delta t^\beta}{\delta s} X^i_\beta + \Omega_{\alpha\beta} t^\beta v^\alpha n^i + \frac{dr}{ds} n^i + r \frac{\delta n^i}{\delta s}. \qquad \ldots(3.4)$$

Substituting the values of $I^i_{\alpha\beta}$ and that of $n^i_{;\alpha}$ eqn. (3.2) simplifies to the following form:

$$\begin{aligned} \nu^i_{;\theta} &= (\Omega_{\theta\beta} t^\beta + r_{;\theta} - \tfrac{1}{2} C_{lhk} n^l n^h X^k_\theta)\, n^i \\ &\quad + [t^\alpha_{;\theta} - \gamma^{\alpha\beta}(\Omega_{\beta\theta} + C_{jhk} X^j_\beta n^h X^k_\theta)]\, X^i_\alpha. \qquad (3.5) \end{aligned}$$

Hence from (3.4) and (3.5) we can write

$$\frac{\delta\nu^i}{\delta s} = \{\Omega_{\theta\beta}t^\beta + r_{;\theta} - \tfrac{1}{2}\, rC_{lhk}n^l n^h X^k_\theta\}\, \nu^\theta n^i$$

$$+ \{t^\alpha_{;\theta} - \gamma^{\alpha\beta}(\Omega_{\beta\theta} + C_{jhk}X^j_\beta n^h X^k_\theta)\}\, \nu^\theta X^i_\alpha . \qquad \ldots(3.6)$$

If $q^i_{\nu|}$ be a vector in the direction of $\frac{\delta\nu^i}{\delta s}$ satisfying the normalizing conditions $g_{ij}q^i_{\nu|}q^j_{\nu|} = 1$ and $R^2_{\nu|} = g_{ij}\frac{\delta\nu^i}{\delta s}\frac{\delta\nu^j}{\delta s}$, eqn. (3.6) can be expressed as

$$\frac{\delta\nu^j}{\delta s} = R_{\nu|}q^j_{\nu|} . \qquad \ldots(3.7)$$

$R_{\nu|}$ can be defined as the absolute curvature of the congruence ν with respect to the curve C. Then defining the normal curvature of the congruence ν with respect to C as the magnitude of the normal component of the derived vector ν divided by the magnitude of the tangent component t^α of ν^i, we write it as $R_{\nu|n}$. Here

$$R^2_{\nu|n} = \{r_{;\theta} - \tfrac{1}{2}\, rC_{jhk}n^j n^h X^k_\theta + \Omega_{\theta\beta}t^\beta\}$$

$$\times \{r_{;\gamma} - \tfrac{1}{2}\, rC_{lmn}n^l n^m X^n_\gamma - \Omega_{\gamma\delta}t^\delta\}\, \nu^\theta \nu^\gamma/(1 - r^2) \qquad \ldots(3.8)$$

the square of the magnitude of t^α being given by, $(1 - r^2) = g_{\alpha\beta}t^\alpha t^\beta$.

We next consider the magnitude of the tangent component of the derived vector ν in (3.6) divided by the magnitude of the tangent component t^α of ν^i. We call this as the geodesic curvature of the congruence ν with respect to C. Denoting this by $R_{\nu|g}$, we have

$$g_{\alpha\beta}\,[t^\alpha_{;\theta} - r\{\gamma^{\alpha\beta}(\Omega_{\beta\theta} + C_{jhk}n^h X^j_\beta X^k_\theta)\}\, R^2_{\nu|g}$$

$$= \frac{\{t^\beta_{;\gamma} - r\gamma^{\beta\delta}(\Omega_{\delta\gamma} + C_{lmn}n^m X^l_\delta X^n_\gamma)\}]\, \nu^\theta \nu^\gamma}{g_{\alpha\delta}t^\alpha t^\delta} . \qquad \ldots(3.9)$$

Simplifying (3.6) with the help of (3.8) and (3.9) we get

$$\frac{\delta\nu^i}{\delta s} = (R_{\nu|n}\, n^i + R_{\nu|g}\, \omega^i)\, t \qquad \ldots(3.10)$$

where (i) t is the magnitude of the tangential component of ν^i, (ii) n is the unit vector of the congruence (ν), and (iii) ω is the unit vector along the geodesic curvature of the congruence (ν), satisfying the condition

$$g_{ij}\omega^i\omega^j = 1.$$

§4. If we put

$$V^\delta_\alpha \equiv t^\delta_{;\alpha} - r\,\{\gamma^{\delta\beta}(\Omega_{\beta\alpha} + C_{jhk}n^h X^j_\beta X^k_\alpha)\} \qquad \ldots(4.1)$$

and

$$W_\alpha \equiv r_{:\alpha} - \tfrac{1}{2}\, r(C_{lmn}n^l n^m X^n_\alpha + \Omega_{\alpha\beta}t^\beta) \qquad ...(4.2)$$

then the tensor derivative of ν^i can be written as follows:

$$\nu^i_{;\alpha} = V^\delta_\alpha \; X^i_\delta + W_\alpha n^i. \qquad ...(4.3)$$

Again taking the tensor derivative of (4.3) and simplifying with the help of (4.1) and (4.2) we get

$$\nu^i_{;\alpha\beta} = [V^\delta_{\alpha;\beta} - W_\alpha\{\gamma^{\delta\theta}(\Omega_{\theta\beta} + C_{jhk}n^h X^j_\theta X^k_\beta)\}]\, X^i_\delta$$
$$+ [V^\delta_\alpha\, \Omega_{\delta\beta} + W_{\alpha;\beta} + \frac{W_\alpha}{2}\, C_{lhk}n^l n^h X^k_\beta\,]\, n^i. \qquad ...(4.4)$$

The operator ∇ is defined as follows :

$$\nabla = X^i_\alpha\, g^{\alpha\beta} \frac{\delta}{\delta x^\beta}$$

where we have assumed that the operator $\frac{\delta}{\delta x^\beta}$ is the symbol for partial δ-differentiation. Thus, we have

$$\frac{\delta \nabla \nu}{\delta s} = (X^i_{;\alpha}\, g^{\alpha\beta} . \nu^j_{;\beta})_{;\gamma}\, \nu^\gamma. \qquad ...(4.5)$$

Laguerre function — The scalar $\mathcal{L}_{\nu|}$ is called the 'generalised Laguerre function' for the direction E.

Defining

$$\mathcal{L}_{\nu|} = - E \,.\, \frac{\delta \nabla \nu}{\delta s} \cdot E$$
$$= -(g_{ij}X^j_\alpha\, \nu^\alpha)\, (X^i_\delta\, g^{\delta\theta}\, \nu^k_{;\theta})_{;\gamma}\, \nu^\gamma (g_{kl}X^l_\beta\, \nu^\beta) \qquad ...(4.6)$$

and then simplifying we get

$$\mathcal{L}_{\nu|} = - [V_{\beta\alpha;\theta} - W_\alpha\{\gamma(\Omega_{\theta\beta} + C_{jkl}X^j_\beta \; X^l_\theta\, n^k)\} + W^\rho_\beta\, C_{\alpha\rho\theta}]\, \nu^\alpha \nu^\beta \nu^\theta. \qquad ...(4.7)$$

Laguerre line — A curve in F_{n-1} for which the tangent vector E satisfies the equation $\mathcal{L}_{\nu|} = 0$ is a 'generalised Laguerre line'.

We now obtain an expression for the Laguerre function in terms of the generalised normal curvature, geodesic curvature of the congruence (ν) with respect to the curve C and geodesic curvature of the curve.

Simplifying (4.6) with the help of (4.1) and (4.2) we have

$$\mathcal{L}_{\nu|} = [t_{\beta;\alpha} - g_{\delta\beta}\{\gamma^{\delta\epsilon}(\Omega_{\alpha\epsilon} + C_{jkl}X^j_\beta X^l_\alpha n^k)\}]\ \nu^\alpha \nu^\beta_{;\gamma}\nu^\gamma$$

$$+ g_{\epsilon\beta}\,[t^\epsilon_{;\alpha} - r\{\gamma^{\tau\epsilon}(\Omega_{\alpha\tau} + C_{jkl}X^j_\tau X^l_\alpha n^k)\}]\ \nu^\alpha \nu^\beta_{;\gamma}\nu^\gamma$$

$$+ g_{\epsilon\beta}\,[t^\epsilon_{;\alpha} - r\{\gamma^{\tau\epsilon}(\Omega_{\alpha\tau} + C_{jkl}X^j_\tau X^l_\alpha n^k)\}]\ \nu^\beta \nu^\alpha_{;\gamma}\nu^\gamma$$

$$- [\tfrac{1}{2}\,r(\Omega_{\alpha\epsilon}t^\epsilon + C_{hkl}n^h n^k X^l_\alpha) - \nu_{;\alpha}][\gamma(\Omega_{\gamma\beta} + C_{jkl}X^j_\beta X^l_\gamma n^k)\,\nu^\alpha\nu^\beta\nu^\gamma]. \quad \ldots(4.8)$$

Let us define the generalised normal curvature $R_{\nu|}$ of the curve relative to the congruence (ν) as follows :

$$R_{\nu|} = [r\gamma(\Omega_{\alpha\beta} + C_{jhk}\,n^h X^j_\beta X^k_\alpha) - t_{\beta;\alpha}]\ \nu^\alpha\nu^\beta. \quad \ldots(4.9)$$

Let θ be the angle between the geodesic curvature vector of the congruence (ν) and the geodesic curvature vector of the curve C. From eqns. (4.8) and (4.9) we have

$$\frac{\delta R_{\nu|}}{\delta s} = [\{r\gamma(\Omega_{\alpha\beta} + C_{jhk}n^h X^j_\beta X^k_\alpha) - t_{\beta;\alpha}\}\ \nu^\alpha\nu^\beta]_{;\gamma}\nu^\gamma. \quad \ldots(4.10)$$

Again, denoting by b the unit vector along the geodesic curvature vector and by R_g the geodesic curvature of the curve C in F_n, we write

$$g_{\epsilon\beta}tR_{\nu|g}a^\epsilon R_g b^\beta = tR_{\nu|g}R_g\cos\theta\cdot \quad \ldots(4.11)$$

Further, let us denote by φ the angle between the curve C and the geodesic curvature vector of the congruence (ν) with respect to another curve $C' : x' = x'(s')$. Since b is orthogonal to ν at a point P, the curves C and C' are orthogonal. Therefore we can write

$$t\bar{R}_{\nu|g}\bar{a}^\epsilon = [t^\epsilon_{;\alpha} - r\{\gamma^{\epsilon\beta}(\Omega_{\beta\alpha} - C_{jhk}n^h X^j_\beta X^k_\alpha)\}]\ b^\alpha \quad \ldots(4.12)$$

wher $\bar{a}^\epsilon$ is the unit vector along the geodesic curvature vector $\bar{R}_{\nu|g}$ of the congruence (ν) with respect to the curve C'. Thus we have

$$t\bar{R}_{\lambda|g}R_g g_{\epsilon\beta}\bar{a}^\epsilon\nu^\beta = t\bar{R}_{\lambda|g}R_g\cos\varphi. \quad \ldots(4.13)$$

Denoting by R_n the normal curvature of the curve C and simplifying eqn. (4.8) with the help of (4.9), (4.10), (4.11), (4.12) and (4.13) we have

$$\mathcal{L}_{\nu|} = tR_g\,[R_{\nu|g}\cos\theta + \bar{R}_{\nu|g}\cos\varphi] + \frac{\delta R_{\nu|}}{\delta s} + tR_{\nu|n}R_n. \quad \ldots(4.14)$$

The generalized Laguerre line satisfies the equation

$$tR_g\begin{bmatrix} R_{\nu|g}\cos\varphi \\ + \bar{R}_{\nu|g}\cos\phi\end{bmatrix} + \frac{\delta R_{\nu|}}{\delta s} + tR_{\nu|n}R_n = 0.$$

§5. The expression (4.14) can also be obtained by a different method given below. Using the symbols and definitions as given by Nirmala (1965) we have

$$E \cdot \nabla_\nu \cdot E = g_{ij} X^i_\alpha \nu^\alpha (X^j_\beta g^{\beta\gamma} \nu^k_{;\gamma}) \cdot g_{kl} X^l_\delta \nu^\delta. \quad \ldots(5.1)$$

Simplifying (5.1) we get

$$E \cdot \nabla_\nu \cdot E = -[r\gamma(\Omega_{\alpha\beta} + C_{jhk} n^h X^j_\beta X^k_\alpha) - t_{\beta;\alpha}] \nu^\alpha \nu^\beta = -R_{\nu|}.$$

That is

$$-R_{\nu|} = E \cdot \nabla_\nu \cdot E. \quad \ldots(5.2)$$

Differentiating (5.2) intrinsically

$$-\frac{\delta R_{\nu|}}{\delta s} = \frac{\delta E}{\delta s} \cdot \nabla_\nu \cdot E + E \frac{\delta \nabla_\nu}{\delta s} \cdot E + E \nabla_\nu \left(\frac{\delta E}{\delta s}\right).$$

Rearranging the terms, we obtain

$$\mathcal{L}_{\nu|} = -E \cdot \frac{\delta \nabla_\nu}{\delta s} \cdot E = \frac{\delta R_{\nu|}}{\delta s} + \frac{\delta E}{\delta s} \cdot \nabla_\nu E + E \nabla_\nu \left(\frac{\delta E}{\delta s}\right). \quad \ldots(5.3)$$

Also, as obtained by Nirmala (1965) we have

$$E \cdot \nabla_E = \frac{\delta E}{\delta s} = R_\nu n + R_g b$$

and

$$E \cdot \nabla_\nu = \frac{\delta \nu}{\delta s} = t(R_{\nu|n} n + R_{\nu|g} a).$$

Thus we have, after some simplifications

$$\frac{\delta E}{\delta s} \cdot \nabla_\nu \cdot E = g_{ij} (R_\nu n^i + R_g b^\alpha X^j_\alpha) X^j_\delta g^{\delta\theta} \nu^k_{;\theta} g_{kl} X^l_\gamma \nu^\gamma$$

$$= t R_g \bar{R}_{\nu|g} \cos \varphi.$$

Simplifying eqn. (5.3) we get the same expression as given by (4.14).

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