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POSITIVE VALUES OF NON-HOMOGENEOUS QUATERNARY QUADRATIC FORMS

V. C. DUMIR AND R. J. HANS-GILL

Department of Mathematics, Panjab University, Chandigarh 160014

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Let $Q(x, y, z, u)$ be a real indefinite quaternary quadratic form of type (1, 3) and determinant $D \neq 0$. Here it is proved that given any real numbers x_0, y_0, z_0, u_0 there exist integers x, y, z, u satisfying

$$0 < Q(x + x_0, y + y_0, z + z_0, u + u_0) \leqslant (16 \mid D \mid)^{1/4}.$$

The critical forms for which equality is needed are also determined.

1. INTRODUCTION

Let $Q(x_1, \ldots, x_n)$ be a real indefinite quadratic form in n variables with signature $(r, n-r)$, $0 < r < n$ and determinant $D \neq 0$. Blaney (1948) has proved that there exist constants Γ independent of Q such that for any real numbers $c_1, \ldots, c_n$ there exist integers $x_1, \ldots, x_n$ satisfying

$$0 < Q(x_1 + c_1, \ldots, x_n + c_n) \leqslant (\Gamma \mid D \mid)^{1/n}.$$

Let $\Gamma_{r,n-r}$ denote the greatest lower bound of all such constants Γ. Davenport and Heilbronn (1947) showed that $\Gamma_{1,1} = 4$. $\Gamma_{2,1} = 4$ was proved independently by Blaney (1950) and Barnes (1961). Dumir (1967) proved that $\Gamma_{1,2} = 8$. Dumir (1968a, b) also showed that $\Gamma_{2,2} = 16$, $\Gamma_{3,1} = \frac{16}{3}$. Hans-Gill and Madhu Raka (1980a) proved that

$$16 \leqslant \Gamma_{1,3} \leqslant 128(2\sqrt{7} - 1)/25 = 21.972 \ldots .$$

The upper bound for $\Gamma_{1,3}$ was used by Hans-Gill and Madhu Raka (1980b, 1981) to determine $\Gamma_{4,1}$ and to prove Watson's conjecture about inhomogeneous minima for the quinary quadratic forms of type (4, 1) or (1, 4). Here we shall prove that $\Gamma_{1,3} = 16$, thereby completing the case $n = 4$.

More precisely we prove:

Theorem 1 — Let $Q(x, y, z, u)$ be an indefinite quaternary quadratic form of type (1, 3) and determinant $D(< 0)$. Then given any real numbers x_0, y_0, z_0, u_0 there exist $(x, y, z, u) \equiv (x_0, y_0, z_0, u_0) \pmod 1$ such that

Dedicated to Professor R. P. Bambah on his 55th Birthday.

$$0 < Q(x, y, z, u) \leqslant (16 \mid D \mid)^{1/4}. \qquad \ldots(1.1)$$

Further, strict inequality holds in (1.1) unless

$$Q \sim \rho(-x^2 - y^2 - z^2 - xz - yz + 2zu) = \rho Q_0, \quad \rho > 0.$$

Also strict inequality holds for Q_0 except when $(x_0, y_0, z_0, u_0) \equiv (0, 0, 0, \frac{1}{2}) \pmod 1$.

When $Q(x, y, z, u)$ is a zero form, Theorem 1 is a special case of a more general theorem of Jackson (1971a). He has also shown that for zero forms (1.1) is satisfied with strict inequality except in the case stated above. So we only need to prove that (1.1) is satisfied with strict inequality for non-zero forms. To prove this we need the following asymmetric inequality for non-zero in homogeneous indefinite ternary forms.

Theorem 2 — Let $Q(x, y, z)$ be an indefinite non-zero ternary quadratic form of type (1, 2) and determinant $D(> 0)$. For $0 < t < \frac{1}{9}$, let $f(t) > \dfrac{256}{25(1-t)^3}$. Then given any real numbers x_0, y_0, z_0 we can find $(x, y, z) \equiv (x_0, y_0, z_0) \pmod 1$ such that

$$t(f(t) \mid D \mid)^{1/3} < Q(x, y, z) < (f(t) \mid D \mid)^{1/3}. \qquad \ldots(1.2)$$

This result is a slight improvement on the corresponding result (Theorem 3) of Hans-Gill and Madhu Raka (1980a). This improvement together with a result of Jackson (1971b) on the positive values of non-zero quaternary quadratic forms of type (3, 1) (see Lemma 1) enable us to complete the proof of Theorem 1. In section 2, we deduce Theorem 1 from Theorem 2 and in section 3 we prove Theorem 2.

2. Quaternary Forms : Proof of Theorem 1

As remarked earlier we need to prove that (1.1) holds with strict inequality for non-zero forms of the type (1, 3). So in this section we shall assume that $Q(x, y, z, u)$ is a non-zero form of type (1, 3) and determinant $D < 0$.

Lemma 1 — There exist integers x, y, z, u such that

$$0 < -Q(x, y, z, u) < (\mid D \mid)^{1/4}.$$

This follows from Jackson (1971b).

Let
$$m = \inf_{\substack{x, y, z, u \text{ integers} \\ Q(x, y, z, u) < 0}} -Q(x, y, z, u)$$

Then $m \geqslant 0$. If $m = 0$, Theorem 1 follows from Theorem 1 of Waston (1960). So we shall suppose that $m > 0$. Let $0 < \epsilon_0 < \frac{3}{4}$. By Lemma 1, there exist integers x_1, y_1, z_1, u_1 such that

$$-Q(x_1, y_1, z_1, u_1) = m/(1-\epsilon) < (\mid D \mid)^{1/4}$$

where $0 \leqslant \epsilon < \epsilon_0$. By the definition of m, we must have g.c.d. $(x_1, y_1, z_1, u_1) = 1$. On replacing Q by an equivalent form we can suppose that

$$-Q(1, 0, 0, 0) = m/(1-\epsilon)$$

and write

$$Q(x, y, z, u) = m(1-\epsilon)^{-1} \{ -(x + hy + gz + ku)^2 + \varphi(y, z, u)\}$$

where $\varphi(y, z, u)$ is an indefinite ternary form of type (1, 2) and determinant

$$| D | (m/(1-\epsilon))^{-4} > 1.$$

Further φ is a non-zero form. Because if y, z, u are integers such that $\varphi(y, z, u) = 0$, then choosing x such that $| x + hy + gz + ku | \leqslant \frac{1}{2}$ we get

$$-m/4(1-\epsilon) \leqslant Q(x, y, z, u) \leqslant 0.$$

This contradicts either the definition of m or the fact that Q is a non-zero form.

Because of homogeneity it suffices to prove:

Theorem A — Let $Q(x, y, z, u) = -(x + hy + gz + ku)^2 + \varphi(y, z, u)$ where $\varphi(y, z, u)$ is a non-zero indefinite ternary quadratic form of type (1, 2) and determinant $| D | > 1$. Let

$$d = (16 | D |)^{1/4}. \quad \ldots(2.1)$$

Then given any real numbers x_0, y_0, z_0, u_0 there exist $(x, y, z, u) \equiv (x_0, y_0, z_0, u_0)$ (mod 1) such that

$$0 < Q(x, y, z, u) < d. \quad \ldots(2.2)$$

Lemma 2 — Let α, β, δ be real numbers with $\delta > 1$. Then for any real number x_0, there exists $x \equiv x_0$ (mod 1) such that

$$0 < -(x+\alpha)^2 + \beta^2 < \delta$$

provided that

$$\tfrac{1}{4} < \beta^2 < \tfrac{1}{4} n^2 + \delta$$

where the integer n is given by $n < \delta \leqslant n + 1$.

This is Lemma 2 of Dumir (1967).

The following lemma is an immediate consequence of Lemma 2.

Lemma 3 — Let $Q(x, y, z, u)$ be as in Theorem A. Suppose we can find $(y, z, u) \equiv (y_0, z_0, u_0)$ (mod 1) such that

$$\tfrac{1}{4} < \varphi(y, z, u) < \tfrac{1}{4} n^2 + d$$

where the integer n is given by $n < d \leqslant n + 1$. Then there exists $x \equiv x_0 \pmod 1$ satisfying (2.2).

Proof of Theorem A

Since $|D| > 1$, $d = (16 |D|)^{1/4} > 2$. Therefore the integer n with $n < d \leqslant n + 1$ satisfies $n \geqslant 2$.

Let $$t = \frac{1}{n^2 + 4d}.$$

Then $$\frac{1}{(n+2)^2} \leqslant t < \frac{1}{n^2 + 4n} \leqslant \frac{1}{12} < \frac{1}{9}.$$

Let $$f(t) = \frac{1}{64\, t^3 |D|} = \frac{1}{4t^3 d^4} = \frac{64\, t}{(1 - n^2 t)^4}.$$

Therefore

$$\frac{1}{(1-t)^3 f(t)} = \frac{(1 - n^2 t)^4}{64\, t(1-t)^3} = \frac{1}{64}\left(\frac{1}{t} - n^2\right)\left(n^2 - \frac{n^2 - 1}{1 - t}\right)^3$$

is a decreasing function of t for $\frac{1}{(n+2)^2} \leqslant t < \frac{1}{n^2 + 4n}$, so that

$$(1-t)^3 f(t) \geqslant \frac{(n+3)^3}{4(n+1)} \geqslant \frac{125}{12} > \frac{256}{25} \text{ for } n \geqslant 2.$$

Thus $$f(t) > \frac{256}{25(1-t)^3}.$$

Hence by Theorem 2, there exist $(y, z, u) \equiv (y_0, z_0, u_0) \pmod 1$ such that

$$\tfrac{1}{4} = t(f(t)\, |D|)^{1/3} < \varphi(y, z, u) < (f(t)\, |D|)^{1/3} = \tfrac{1}{4} n^2 + d.$$

Now Theorem A follows from Lemma 3. This completes the proof of Theorem 1.

3. Ternary Forms : Proof of Theorem 2

Let $Q(x, y, z)$ be an indefinite non-zero ternary quadratic form of type (1, 2) and determinant $D > 0$. We need the following result of Oppenheim (1953) on the negative values of ternary quadratic forms of type (1, 2).

Lemma 4 — There exist integers u, v, w such that

$$0 < -Q(u, v, w) \leqslant (4D)^{1/3}.$$

Let $$m' = \inf_{\substack{u,\, v,\, w \text{ integers} \\ Q(u, v, w) < 0}} -Q(u, v, w)$$

Then $m' \geqslant 0$. If $m' = 0$, then (1.2) follows from Theorem 1 of Watson (1960). So we can suppose that $m' > 0$.

Let $\epsilon_0 > 0$ be sufficiently small. By Lemma 4, there exist integers u, v, w such that

$$-Q(u, v, w) = \frac{m'}{1-\epsilon} \leqslant (4D)^{1/3},$$

where $0 \leqslant \epsilon < \epsilon_0$. Also g.c.d $(u, v, w) = 1$ by the definition of m'. Applying a suitable unimodular transformation to Q we can suppose that

$$-Q(1, 0, 0) = \frac{m'}{1-\epsilon}$$

and write

$$Q(x, y, z,) = \frac{m'}{1-\epsilon}\{-(x + hy + gz)^2 + \varphi(y, z)\}$$

where $\varphi(y, z)$ is an indefinite quadratic form of discriminant

$$\Delta^2 = 4D \Big/ \left(\frac{m'}{1-\epsilon}\right)^3 \geqslant 1.$$

Because of homogeneity it suffices to prove:

Theorem B — Let $Q(x, y, z) = -(x + hy + gz)^2 + \varphi(y, z)$ where $\varphi(y, z)$ is an indefinite binary quadratic form with discriminant

$$\Delta^2 = 4D \geqslant 1. \qquad \text{...(3.1)}$$

Further, for integers u, v, w either

$$-Q(u, v, w) \geqslant 1 - \epsilon \quad \text{or} \quad -Q(u, v, w) < 0. \qquad \text{...(3.2)}$$

For $0 < t < \frac{1}{9}$, let

$$f(t) > \frac{256}{25(1-t)^3} \quad \text{and} \quad d = (f(t)\, D)^{1/3}. \qquad \text{...(3.3)}$$

Then given any real numbers x_0, y_0, z_0 there exist $(x, y, z) \equiv (x_0, y_0, z_0) \pmod 1$ such that

$$td < Q(x, y, z) < d. \qquad \text{...(3.4)}$$

We shall use the following notation throughout this section. For fixed t and d, n is the integer defined by

$$n < (1 - t)\, d \leqslant n + 1. \qquad \text{...(3.5)}$$

Since $(1 - t)^3 d^3 = (1 - t)^3 f(t) D > 64/25 > (4/3)^3$ it follows that $n \geqslant 1$, also

$$(1 - t)\, d > \tfrac{4}{3}. \qquad \text{...(3.6)}$$

Let $\sigma = \frac{1}{4} + td, \quad \rho = \frac{1}{4} n^2 + d.$ ...(3.7)

Lemma 5 — Let $Q(x, y, z,)$ be as in Theorem B. Suppose $(y, z) \equiv (y_0, z_0)$ (mod 1) are such that

$$\sigma < \varphi(y, z) < \rho. \qquad \text{...(3.8)}$$

Then there exists $x \equiv x_0$ (mod 1) such that (3.4) is satisfied.

This result follows at once from Lemma 2 by taking

$$\alpha = hy + gz, \ \beta^2 = \varphi(y, z) - td, \ \delta = (1 - t)\, d$$

and noting that $\delta > \frac{4}{3}$ by (3.6).

Lemma 6 — Suppose $Q(x, y, z)$ satisfies the conditions of Theorem B. Then for integers y, z

either $\quad \varphi(y, z) \leqslant - \frac{3}{4} + \epsilon \quad$ or $\quad \varphi(y, z) > 0.$

PROOF: Suppose y, z are integers for which $- \frac{3}{4} + \epsilon < \varphi(y, z) \leqslant 0$. Choose an integer x such that $| x + hy + gz | \leqslant \frac{1}{2}$. Then $0 \leqslant -Q(x, y, z) < 1 - \epsilon$, which contradicts (3.2). This proves the lemma.

To prove Theorem B, it is enough to prove that (3.8) is satisfied. We shall need some inequalities which we prove in the following lemma.

Lemma 7 — Let $n, \ \rho, \ \sigma$ be as defined in (3.5) and (3.7). Then for all $n \geqslant 1$ we have

$$\tfrac{9}{8} \Delta^2 < (\rho - \sigma)^2 \qquad \text{...(3.9)}$$

$$4\sigma < \rho \qquad \text{...(3.10)}$$

$$4\Delta^2 < (\rho - \sigma)(9\rho - 25\sigma) \qquad \text{...(3.11)}$$

and $\quad$ for $n = 2, \ \frac{4}{3} \Delta^2 < (\rho - \sigma)^2.$ ...(3.12)

PROOF: Since $\Delta^2 = 4D = 4d^3/f(t)$: and $\rho - \sigma = \frac{1}{4}\{n^2 - 1 + 4(1 - t)\, d\}$, we have

$$\left(\frac{\rho - \sigma}{\Delta}\right)^2 = \frac{f(t)}{64} \frac{\{n^2 - 1 + 4(1 - t)\, d\}^2}{d^3}$$

$$= \frac{f(t)}{64} \frac{1}{d} \left(\frac{n^2 - 1}{d} + 4(1 - t)\right)^2$$

which is a decreasing function of d. Since $d \leqslant (n + 1)/(1 - t)$ [by (3.5)], we have

$$\frac{(\rho - \sigma^2)}{\Delta^2} \geqslant \frac{f(t)}{64} (1 - t)^3 \frac{(n + 3)^2}{(n + 1)} > \frac{4}{25} \frac{(n + 3)^2}{n + 1} \quad \text{[by (3.3)]}.$$

The right-hand side is an increasing function of n, so that

$$\frac{(\rho - \sigma)^2}{\Delta^2} > \begin{cases} \frac{9}{8} \text{ for } n \geqslant 1 \\ \frac{4}{3} \text{ if } n = 2. \end{cases}$$

This proves (3.9) and (3.12).

Since $n \geqslant 1$, and $(1 - t)\, d > \frac{4}{3}$ by (3.6),

$$\rho - 4\sigma = \frac{n^2}{4} - 1 + (1 - 4t)\, d > -\frac{3}{4} + \frac{4(1 - 4t)}{3(1 - t)} > 0, \text{ for } t < \tfrac{1}{9}.$$

This proves (3.10). The inequality (3.11) easily follows from (3.9) and (3.10).

By the Markoff Chain Theorem, if $\varphi(y, z) \not\sim b(y^2 + yz - z^2)$ then there exist integers v, w not both zero with $(v, w) = 1$ such that $a = \varphi(v, w)$ and $|a| \leqslant \Delta/\sqrt{8}$. In fact one can suppose that $|a| \leqslant \Delta/3$, unless $\varphi(y, z)$ is a Markoff form. A Markoff form represents both a and $-a$, so for such forms, we can assume that $a > 0$. Also if $a = -b < 0$, then by Lemma 6, we have $b \geqslant \frac{3}{4} - \epsilon$. Since φ is a non-zero form we distinguish the following cases:

(I) $0 < a \leqslant \Delta/\sqrt{8}$;

(II) $a = -b,\ \frac{3}{4} - \epsilon \leqslant b \leqslant \Delta/3$;

or $\qquad \varphi(y, z) \sim a(y^2 + yz - z^2),\ a = -b < 0$;

3.1. *Case I*

In this case on replacing $\varphi(y, z)$ by an equivalent form we can suppose that

$$\varphi(y, z) = a(y + fz)^2 - \frac{\Delta^2}{4a} z^2.$$

The following Lemma follows from Lemma 6 of Dumir (1968a).

Lemma 8 — Let α, β and δ be real numbers with $\delta > 1$. Then given any real number x_0, there exists $x \equiv x_0 \pmod 1$ satisfying

$$0 < (x + \alpha)^2 - \beta^2 < \delta$$

provided $\beta^2 < \frac{1}{4} p^2$, where the integer p is given by $p < \delta \leqslant p + 1$.

From this we deduce the following result.

Lemma 9 — Let $\varphi(y, z) = a(y + fz)^2 - (\Delta^2/4a)\, z^2$, $a > 0$. Let ρ', σ' be real numbers such that $0 < \sigma' < \rho'$ and $a < \rho' - \sigma'$. Let p be the postive integer determined by

$$p < \frac{\rho' - \sigma'}{a} \leqslant p + 1. \qquad \ldots(3.13)$$

Suppose that

$$h(a) = 4p^2a^2 - 16\sigma' a > \Delta^2. \qquad \ldots(3.14)$$

Then given any real numbers (y_0, z_0) there exist $(y, z) \equiv (y_0, z_0) \pmod 1$ such that

$$\sigma' < \varphi(y, z) < \rho'. \qquad \ldots(3.15)$$

PROOF : (3.15) can be rewritten as

$$0 < (y + fz)^2 - \left\{(\Delta^2/4a^2)\, z^2 + \frac{\sigma'}{a}\right\} < \frac{\rho' - \sigma'}{a}. \qquad \ldots(3.16)$$

Choose $z \equiv z_0 \pmod 1$ satisfying $|z| \leqslant \frac{1}{2}$. Then

$$\frac{\Delta^2}{4a^2} z^2 + \frac{\sigma'}{a} \leqslant \frac{\Delta^2}{16a^2} + \frac{\sigma'}{a}$$

$$< \left(\frac{p}{2}\right)^2, \text{ by (3.14).}$$

Now the solubility of (3.16) follows from Lemma 8 with

$$\alpha = fz,\ \beta^2 = \frac{\Delta^2}{4a^2} z^2 + \frac{\sigma'}{a} \ \text{ and } \ \delta = \frac{\rho' - \sigma'}{a}.$$

Lemma 10 — If $0 < a \leqslant \Delta/\sqrt{8}$, then (3.8) is satisfied.

PROOF : Let n, ρ, σ be as defined in (3.5) and (3.7). We shall apply Lemma 9 with $\rho' = \rho$, $\sigma' = \sigma$. Now

$$\left(\frac{\rho - \sigma}{a}\right)^2 \geqslant \frac{8(\rho - \sigma)^2}{\Delta^2} > 9, \text{ by (3.9).}$$

Let p be the integer given by $p < \dfrac{\rho - \sigma}{a} \leqslant p + 1$. Then $p \geqslant 3$. It follows easily from (3.10) that $h(a)$ is an increasing function of a, for fixed p. Therefore

$$h(a) \geqslant h\left(\frac{\rho - \sigma}{p + 1}\right) = \frac{4p^2(\rho - \sigma)^2}{(p + 1)^2} - \frac{16\sigma(\rho - \sigma)}{p + 1} = g(p), \text{ say.}$$

It is easy to see that $g(p)$ is an increasing function of p, so that

$$h(a) \geqslant g(3) = \tfrac{1}{4}(\rho - \sigma)(9\rho - 25\sigma) > \Delta^2 \text{ by (3.11).}$$

Thus the hypothesis of Lemma 9 is verified, so that (3.15) and hence (3.8) is satisfied.

3.2. *Case II*

In this case, on replacing $\varphi(y, z)$ by an equivalent form we can suppose that

$$\varphi(y, z) = -b(y + fz)^2 + (\Delta^2/4b)\, z^2, \ b > 0.$$

we need a lemma similar to Lemma 9.

Lemma 11 — Let $\varphi(y, z) = -b(y + fz)^2 + (\Delta^2/4b)z^2$, $b > 0$. Let ρ', σ' be real numbers such that $0 < \sigma' < \rho'$ and $b < \rho' - \sigma'$. Let the integers q and m be determined by

$$q < \frac{\rho' - \sigma'}{b} \leqslant q + 1 \qquad \ldots(3.17)$$

and
$$2\sqrt{b^2 + 4b\sigma'} < m\,\Delta \leqslant \Delta + 2\sqrt{b^2 + 4b\sigma'}. \qquad \ldots(3.18)$$

Suppose that

$$(m + 1)^2\,\Delta^2 < 4q^2b^2 + 16b\rho' \qquad \ldots(3.19)$$

Then given any real numbers y_0, z_0 there exist $(y, z) \equiv (y_0, z_0) \pmod 1$ such that

$$\sigma' < \varphi(y, z) < \rho'. \qquad \ldots(3.20)$$

PROOF : (3.20) can be written as

$$0 < -(y + fz)^2 + \left(\frac{\Delta^2}{4b^2}\,z^2 - \frac{\sigma'}{b}\right) < \frac{\rho' - \sigma'}{b}. \qquad \ldots(3.21)$$

We can choose $z \equiv z_0 \pmod 1$ such that $\frac{m}{2} \leqslant |\,z\,| \leqslant \frac{m + 1}{2}$. Then (3.18) and (3.19) imply

$$\frac{b^2 + 4b\sigma'}{\Delta^2} < z^2 < \frac{q^2b^2 + 4b\rho'}{\Delta^2}$$

or
$$\tfrac{1}{4} < \frac{\Delta^2 z^2}{4b^2} - \frac{\sigma'}{b} < \tfrac{1}{4}\,q^2 + \frac{\rho' - \sigma'}{b}.$$

Now the result follows from Lemma 2 on taking $\alpha = fz$,

$$\beta^2 = \frac{\Delta^2 z^2}{4b^2} - \frac{\sigma'}{b} \quad \text{and} \quad \delta = \frac{\rho' - \sigma'}{b}.$$

Lemma 12 — If $a = -b$, $\frac{3}{4} - \epsilon < b \leqslant \Delta/3$, then (3.8) is satisfied.

PROOF : Let n, ρ, σ be as defined in (3.5) and (3.7). Take $\rho' = \rho$ and $\sigma' = \sigma$ in Lemma 11. Let q, m be as defined in (3.17) and (3.18). Here $q \geqslant 3$, because $b \leqslant \Delta/3 < (\rho - \sigma)/3$ by (3.9). We now verify that for each m, the condition (3.19) of Lemma 11 is satisfied.

Case (i) : $m \geqslant 3$

By (3.18), $(m - 1)\,\Delta \leqslant 2\sqrt{b^2 + 4b\sigma}$. Therefore,

$$\begin{aligned}(m+1)^2\Delta^2 &\leqslant 4(m+1)^2(b^2+4b\sigma)/(m-1)^2\\ &\leqslant 16(b^2+4b\sigma) \quad (\text{for } m \geqslant 3)\\ &< 16b^2+16b\rho \ \{\text{since } 4\sigma<\rho \text{ by } (3.10)\}\\ &\leqslant 4q^2b^2+16b\rho, \quad \text{for } q\geqslant 2.\end{aligned}$$

Case (ii) : $m = 1$

Since $q \geqslant \frac{\rho-\sigma}{b} - 1 > 0$, we get

$$\begin{aligned}4q^2b^2+16b\rho &\geqslant 4\,[(\rho-\sigma-b)^2+4b\rho]\\ &= 4\,[(\rho-\sigma)^2+2b(\rho+\sigma)+b^2]\\ &> 4\,(\rho-\sigma)^2 > 4\,\Delta^2 = (m+1)^2\Delta^2 \quad (\text{by } 3.9).\end{aligned}$$

Here we notice that the discussion of cases (i) and (ii) is also valid for $q \geqslant 2$ (We shall use this in the next lemma).

Case (iii) : $m = 2$

Since $\Delta \geqslant 3b > 3\,(3/4-\epsilon)$, we have

$$(1-t)^3d^3 = \frac{(1-t)^3 f(t)\,\Delta^2}{4} > \frac{64}{25}\cdot q(3/4-\epsilon)^2 > 8$$

if ϵ is sufficiently small. Thus $(1-t)\,d > 2$ and therefore $n \geqslant 2$.

Since $m = 2$, by (3.18) we have

$$\frac{\Delta}{2} \leqslant \sqrt{b^2+4b\sigma} < \Delta \qquad \ldots(3.22)$$

and the condition (3.19), which is to be verified, becomes

$$\tfrac{9}{4}\,\Delta^2 < q^2b^2 + 4b\rho. \qquad (3.23)$$

We now distinguish subcases according to the values of n and q.

Subcase 1: $n \geqslant 3$ — By (3.22),

$$\begin{aligned}\tfrac{9}{4}\,\Delta^2 &\leqslant 9b^2+36b\sigma\\ &= 9b^2+4b\rho-4b(\rho-9\sigma)\\ &< q^2b^2+4b\rho,\end{aligned}$$

because $q \geqslant 3$ and $\rho - 9\sigma = \frac{1}{4}\{n^2-9+4(1-9\,t)\,d\} > 0$ for $n \geqslant 3$ and $t < \frac{1}{9}$.

Subcase 2 : $n = 2,\ q \geqslant 4$ — By (3.22),

$$\tfrac{9}{4}\,\Delta^2 \leqslant 9b^2 + 36b\sigma$$

(*equation continued on p.* 824)

$$= 16b^2 + 4b\rho - b[7b + 4(\rho - 9\sigma)]$$
$$< q^2b^2 + 4b\rho,$$

since

$$7b + 4(\rho - 9\sigma) = 7b - 5 + 4(1 - 9t)\,d$$
$$\geqslant 7(3/4 - \epsilon) - 5 > 0$$

for ϵ sufficiently small.

Subcase 3 : $n = 2$, $q = 3$ — Since $q = 3$, by (3.17) we have $b \geqslant \frac{\rho - \sigma}{4}$, so that

$$q^2b^2 + 4b\rho = 9b^2 + 4b\rho$$
$$= (b^2 + 4b\sigma) + 8b^2 + 4b(\rho - \sigma)$$
$$\geqslant \frac{\Delta^2}{4} + \frac{(\rho - \sigma)^2}{2} + (\rho - \sigma)^2 \left(\text{using (3.22) and } b \geqslant \frac{\rho - \sigma}{4}\right)$$
$$= \tfrac{1}{4}\,\Delta^2 + \tfrac{3}{2}\,(\rho - \sigma)^2$$
$$> \tfrac{9}{4}\,\Delta^2 \text{ by (3.12).}$$

Thus the condition (3.19) of Lemma 11 is satisfied in each case and it follows that (3.8) is soluble.

Lemma 13 — If $\varphi(y, z) \sim -b(y^2 + yz - z^2)$, $b > 0$, then (3.8) is satified.

PROOF : Here we proceed as in the proof of Lemma 12. Let n, ρ, σ, q and m be as defined in (3.5), (3.7), (3.17) and (3.18) respectively. By (3.6), $n \geqslant 1$. Since $b = \Delta/\sqrt{5} < \frac{\rho - \sigma}{2}$, by (3.9), we have also $q \geqslant 2$.

To apply Lemma 11, we have to verify that condition (3.19) is satisfied. If $m \geqslant 3$ or $m = 1$, the proof is the same as in cases (i) and (ii) of Lemma 12. Thus it remains to consider the case $m = 2$ only. Here,

$$4q^2b^2 + 16b\rho \geqslant 16b(b + \rho)$$
$$> 16\,\Delta/\sqrt{5}\,(\Delta/\sqrt{5} + \Delta) \text{ (since } \rho > \rho - \sigma > \Delta, \text{ by (3.9))}$$
$$= \frac{16(1 + \sqrt{5})}{5}\,\Delta^2$$
$$> 9\,\Delta^2 = (m + 1)^2\,\Delta^2.$$

Thus (3.19) is satisfied and the lemma follows from Lemma 11.

Theorem B follows from Lemmas 5, 10, 12 and 13 and the proof of Theorem 2 is completed.

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