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CONCERNING SOME PROPERTIES OF THE *H*-FUNCTION OF SEVERAL VARIABLES

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Parameters of the *H*-function of several variables are classified in two categories and are termed as the 'first type' and the 'second type'. This notion is fully exploited in obtaining a set of $(r + 2)$-term and three-term contiguous function relations. A scheme is also described that enables one to see all such possible contiguous relations. Methods involving three-term contiguous function relations, derivatives, partial derivatives and the Mellin transformation are then used to discover certain finite summation formulas. An application has been given that identifies the *H*-function with a generalized probability density function.

1. INTRODUCTION

As a natural generalization of the generalized Lauricella function (Srivastava and Daoust 1969) of several complex variables, recently, Srivastava and Panda (1976a, b) defined the *H*-function of several complex variables $z_1, \ldots, z_r$ in terms of multiple contour integral of Mellin-Barnes type:

$$H^{0,\lambda:(\mu',\nu');\ldots;(\mu^{(r)},\nu^{(r)})}_{A,C:[B',D'];\cdots;[B^{(r)},D^{(r)}]}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}];\\ [(c):\psi',\ldots,\psi^{(r)}]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ z_1,\ldots,z_r\right)$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\Psi(s_1,\ldots,s_r)\,\Phi_1(s_1)\ldots\Phi_r(s_r)\,Z_1^{s_1}\ldots Z_r^{s_r}\,ds_1\ldots ds_r, \qquad \ldots(1.1)$$

$$\omega = \sqrt{-1}$$

where

$$\Phi_i(s_i) = \frac{\prod_{j=1}^{\mu^{(i)}}\Gamma[d_j^{(i)} - \delta_j^{(i)}s_i]\prod_{j=1}^{\nu^{(i)}}\Gamma[1 - b_j^{(i)} + \phi_j^{(i)}s_i]}{\prod_{j=\mu^{(i)}+1}^{D^{(i)}}\Gamma[1 - d_j^{(i)} + \delta_j^{(i)}s_i]\prod_{j=\nu^{(i)}+1}^{B^{(i)}}\Gamma[b_j^{(i)} - \phi_j^{(i)}s_i]}, \qquad \ldots(1.2)$$

$$i = 1, \ldots, r;$$

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$$\Psi(s_i, \ldots, s_1) = \frac{\prod_{j=1}^{\lambda} \Gamma\left[1 - a_j + \sum_{i=1}^{r} \theta_j^{(i)} s_i\right]}{\prod_{j=\lambda+1}^{A} \Gamma\left[a_j - \sum_{i=1}^{r} \theta_j^{(i)} s_i\right] \prod_{j=1}^{C} \Gamma\left[1 - c_j + \sum_{i=1}^{r} \psi_j^{(i)} s_i\right]}, \quad \ldots(1.3)$$

an empty product is interpreted as unity, the coefficients $\theta_j^{(i)}$, $j = 1, \ldots, A$; $\phi_j^{(i)}$, $j = 1, \ldots, B^{(i)}$; $\psi_j^{(i)}$, $j = 1, \ldots, C$; $\delta_j^{(i)}$, $j = 1, \ldots, D^{(i)}$; and $i = 1, \ldots, r$ are positive real numbers, and λ, $\mu^{(i)}$, $\nu^{(i)}$, A, $B^{(i)}$, C, $D^{(i)}$ are integers such that $0 \leqslant \lambda \leqslant A$, $0 \leqslant \mu^{(i)} \leqslant D^{(i)}$, $C \geqslant 0$ and $0 \leqslant \nu^{(i)} \leqslant B^{(i)}$, $i = 1, \ldots, r$. The contour L_i in the complex s_i-plane is of the Mellin-Barnes type which runs from $-\omega\infty$ to $\omega\infty$ with indentations, if necessary, in such a manner that all the poles of $\Gamma[d_j^{(i)} - \delta_j^{(i)} s^i]$, $j = 1, \ldots, \mu^{(i)}$, are to the right and those of $\Gamma[1 - b_j^{(i)} + \phi_j^{(i)} s^i]$, $j = 1, \ldots, \nu^{(i)}$ and $\Gamma\left[1 - a_j + \sum_{i=1}^{r} \theta_j^{(i)} s_i\right]$, $j = 1, \ldots, \lambda$, to the left of L_i, the various parameters being so restricted that these poles are all simple and none of them coincide; and with the points $z_i = 0$, $i = 1, \ldots, r$, being tacitly excluded. The multiple integral in (1.1) converges absolutely if

$$|\arg(z_i)| < \tfrac{1}{2}\pi\Delta_i^*, \quad i = 1, \ldots, r \quad \ldots(1.4)$$

where

$$\Delta_i^* = \sum_{j=1}^{\lambda} \theta_j^{(i)} - \sum_{j=\lambda+1}^{A} \theta_j^{(i)} + \sum_{j=1}^{\nu^{(i)}} \phi_j^{(i)} - \sum_{j=\nu^{(i)}+1}^{B^{(i)}} \phi_j^{(i)} + \sum_{j=1}^{\mu^{(i)}} \delta_j^{(i)} - \sum_{j=1}^{C} \psi_j^{(i)} - \sum_{j=\mu^{(i)}+1}^{D^{(i)}} \delta_j^{(i)} > 0, \quad i = 1, \ldots, r. \quad \ldots(1.5)$$

Throughout the present paper we use the abbreviation (a) to denote the sequence of A parameters $a_1, \ldots, a_A$; with similar interpretations for other parameters. We shall designate the parameters from the sequences (a) and (c) as the 'first type' and from the sequences $(b^{(i)})$ and $(d^{(i)})$, $i = 1, \ldots, r$, as the 'second type'. To simplify things further we shall also call any two 'second type' parameters to be of the 'same kind' if they have the same superscripts, otherwise they will be termed as being of 'different kind'.

2. Contiguous Function Relations

For the sake of convenience, we merely write H for the function as given in the defining relation (1.1); then we use $H[a_j - 1]$ to denote the contiguous function in which a_j is replaced by $(a_j - 1)$ but with all other parameters left unchanged. We notice that the replacement of a_j by $(a_j - 1)$ and application of the formula $\Gamma(1 + x) = x\Gamma(x)$ introduces the multiplier

$$[1 - a_j + \sum_{i=1}^{r} \theta_j^{(i)} s_i]$$

in the contour integral format of H provided $1 \leqslant j \leqslant \lambda$, on the other hand if $\lambda + 1 \leqslant j \leqslant A$, the multiplying factor introduced would be $[a_j - 1 - \sum_{i=1}^{r} \theta_j^{(i)} s_i]$. If we denote $Q(s_1, \ldots, s_r)$ for the gamma product in the integrand of the defining integral (1.1), viz.,

$$Q(s_1, \ldots, s_r) = \Phi_1(s_1), \ldots, \Phi_r(s_r)\ \Psi(s_1, \ldots, s_r), \qquad \ldots(2.1)$$

and denote $Q_{(a_j-1)}(s_1, \ldots, s_r)$ for the expression in which a_j is replaced by $a_j - 1$, then

$$Q_{(a_j-1)}(s_1, \ldots, s_r) = \begin{cases} [1 - a_j + \sum_{i=1}^{r} \theta_j^{(i)} s_i]\ Q(s_1, \ldots, s_r);\ 1 \leqslant j \leqslant \lambda \\ [a_j - 1 - \sum_{i=1}^{r} \theta_j^{(i)} s_i]\ Q(s_1, \ldots, s_r);\ \lambda + 1 \leqslant j \leqslant A, \end{cases} \qquad \ldots(2.2)$$

in the sequel, it also follows that

$$Q_{(b_j^{(i)}-1)}(s_1, \ldots, s_r) = \begin{cases} [1 - b_j^{(i)} + \phi_j^{(i)} s_i]\ Q(s_1, \ldots, s_r);\ 1 \leqslant j \leqslant \nu^{(i)} \\ [b_j^{(i)} - 1 - \phi_j^{(i)} s_i]\ Q(s_1, \ldots, s_r);\ \nu^{(i)} + 1 \leqslant j \leqslant B^{(i)}, \end{cases} \qquad \ldots(2.3)$$

$$Q_{(c_j+1)}(s_1, \ldots, s_r) = \left[-c_j + \sum_{i=1}^{r} \psi_j^{(i)} s_i\right] Q(s_1, \ldots, s_r);\ 1 \leqslant j \leqslant C, \qquad \ldots(2.4)$$

$$Q_{(d_j^{(i)}+1)}(s_1, \ldots, s_r) = \begin{cases} [d_j^{(i)} - \delta_j^{(i)} s_i]\ Q(s_1, \ldots, s_r);\ 1 \leqslant j \leqslant \mu^{(i)} \\ [-d_j^{(i)} + \delta_j^{(i)} s_i]\ Q(s_1, \ldots, s_r);\ \mu^{(i)} + 1 \leqslant j \leqslant D^{(i)}, \end{cases} \qquad i = 1, \ldots, r. \qquad \ldots(2.5)$$

Since no contiguous function relation involving any parameter of the 'first type' having less than $(r + 2)$ terms can exist, in order to form a $(r + 2)$ term relation comprising $(r + 1)$ contiguous functions with undetermined coefficients in the form

$$P_1 H[a_1 - 1] + \ldots + P_{r+1} H[a_{r+1} - 1] = PH \qquad \ldots(2.6)$$

we assume in the first case that $(r + 1) \leqslant \lambda$ and require that the expression

$$P_1\left[1 - a_1 + \sum_{i=1}^{r} \theta_1^{(i)} s_i\right] + \ldots + P_{r+1}\left[1 - a_{r+1} + \sum_{i=1}^{r} \theta_{r+1}^{(i)} s_i\right] = P \qquad \ldots(2.7)$$

which is obtainable from (2.2) and (2.6), is an identity in $s_1, \ldots, s_r$. This yields $(r+1)$ equations in $(r+2)$ undetermined coefficients. Hence $P_1, \ldots, P_{r+1}, P$ can be evaluated and substitution in (2.6) gives the contiguous relation

$$\begin{vmatrix} H[a_1-1] \ldots H[a_\lambda-1] & H[a_{\lambda+1}-1] \ldots H[a_{r+1}-1] \\ \theta_1' \ldots \theta_\lambda' & \theta_{\lambda+1}' \ldots \theta_{r+1}' \\ \vdots \quad \vdots & \vdots \quad \vdots \\ \theta_1^{(r)} \ldots \theta_\lambda^{(r)} & \theta_{\lambda+1}^{(r)} \ldots \theta_{r+1}^{(r)} \end{vmatrix}$$

$$= \begin{vmatrix} 1-a_1 \ldots 1-a_\lambda & 1-a_{\lambda+1} \ldots 1-a_{r+1} \\ \theta_1' \ldots \theta_\lambda' & \theta_{\lambda+1}' \ldots \theta_{r+1}' \\ \vdots \quad \vdots & \vdots \quad \vdots \\ \theta_1^{(r)} \quad \theta_\lambda^{(r)} & \theta_{\lambda+1}^{(r)} \quad \theta_{r+1}^{(r)} \end{vmatrix} H. \qquad \ldots(2.8)$$

If however, $1 \leqslant \lambda < r+1$, one may also visualize from (2.8) in the light of (2.2), a corresponding contiguous relation if for $\lambda+1 \leqslant k < r+1$, $1-a_k$ and $\theta_k^{(i)}$ are replaced by $a_k - 1$ and $-\theta_k^{(i)}$, $i = 1, \ldots r$, respectively.

Contiguous relations involving parameters from the sequences (a), (c), $(b^{(i)})$, $(d^{(i)})$ $i = 1, \ldots, r$, which include both the 'first type and the second type' of parameters, and their various combinations can be written. As an example, for

$$1 \leqslant k \leqslant \lambda, \quad 1 \leqslant p \leqslant C, \quad 1 \leqslant m \leqslant \nu^{(i)}, \quad 1 \leqslant (r+1-k-p-m) \leqslant \mu^{(i)}$$

a contiguous relation between the parameters from the sequences (a), (c), $(b^{(i)})$ and $(d^{(i)})$ is given by

$$\begin{vmatrix} H[a_1-1] \ldots H[a_k-1] & H[c_1+1] \ldots H[c_p+1] \\ \theta_1' \quad \ldots \quad \theta_k' & \psi_1' \quad \ldots \quad \psi_p' \\ \cdot \quad \ldots \quad \cdot & \cdot \quad \ldots \quad \cdot \\ \theta_1^{(i)} \quad \ldots \quad \theta_k^{(i)} & \psi_1^{(i)} \quad \ldots \quad \psi_p^{(i)} \\ \cdot \quad \ldots \quad \cdot & \cdot \quad \ldots \quad \cdot \\ \theta_1^{(r)} \quad \ldots \quad \theta_k^{(r)} & \psi_1^{(r)} \quad \ldots \quad \psi_p^{(r)} \end{vmatrix}$$

$$\begin{vmatrix} H[b_1^{(i)}-1] \ldots H[b_m^{(i)}-1] & H[d_1^{(i)}+1] \ldots H[d_{r+1-k-p-m}^{(i)}+1] \\ 0 \quad \ldots \quad 0 & 0 \quad \ldots \quad 0 \\ \cdot \quad \ldots \quad \cdot & \cdot \quad \ldots \quad \cdot \\ \phi_1^{(i)} \quad \ldots \quad \phi_m^{(i)} & -\delta_1^{(i)} \quad \ldots \quad -\delta_{r+1-k-p-m}^{(i)} \\ \cdot \quad \ldots \quad \cdot & \cdot \quad \ldots \quad \cdot \\ 0 \quad \ldots \quad 0 & 0 \quad \ldots \quad 0 \end{vmatrix} =$$

(equation continued on p. 830)

$$= \begin{vmatrix} 1-a_1 & \dots & 1-a_k & -c_1 & \dots & -c_p \\ \theta_1' & \dots & \theta_k' & \psi_1' & \dots & \psi_p' \\ \cdot & \dots & \cdot & \cdot & \dots & \cdot \\ \theta_1^{(i)} & \dots & \theta_k^{(i)} & \psi_1^{(i)} & \dots & \psi_p^{(i)} \\ \cdot & \dots & \cdot & \cdot & \dots & \cdot \\ \theta_1^{(r)} & \dots & \theta_k^{(r)} & \psi_1^{(r)} & \dots & \psi_p^{(r)} \end{vmatrix}$$

$$\begin{vmatrix} 1-b_1^{(i)} & \dots & 1-b_m^{(i)} & d_1^{(i)} & \dots & d_{r+1-k-p-m}^{(i)} \\ 0 & \dots & 0 & 0 & \dots & 0 \\ \cdot & \dots & \cdot & \cdot & \dots & \cdot \\ \phi_1^{(i)} & \dots & \phi_m^{(i)} & -\delta_1^{(i)} & \dots & -\delta_{r+1-k-p-m}^{(i)} \\ \cdot & \dots & \cdot & \cdot & \dots & \cdot \\ 0 & \dots & 0 & 0 & \dots & 0 \end{vmatrix} H. \qquad \dots(2.9)$$

It is interesting to remark that all such contiguous function relations, in which more than one parameter from the same sequence occurs, become identically zero whenever H-function is so specialized that it corresponds to multiple G-function or Lauricella's r-ple functions. This may in particular can be visualized from (2.8) and (2.9). In contrast, however, the most significant form for which this does not hold, is that which involves at the most one parameter from one sequence. This is manifested by the following example:

$$\begin{vmatrix} H[a_j-1] & H[b_{k_1}'-1] & \dots & H[b_{k_i}^{(i)}-1] & \dots & H[b_{k_r}^{(r)}-1] \\ \theta_j' & \phi_{k_1}' & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ \theta_j^{(i)} & 0 & \dots & \phi_{k_i}^{(i)} & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ \theta_j^{(r)} & 0 & \dots & 0 & \dots & \phi_{k_r}^{(r)} \end{vmatrix}$$

$$= \begin{vmatrix} 1-a_j & 1-b_{k_1}' & \dots & 1-b_{k_i}^{(i)} & \dots & 1-b_{k_r}^{(r)} \\ \theta_j' & \phi_{k_1}' & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ \theta_j^{(i)} & 0 & \dots & \phi_{k_i}^{(i)} & \dots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ \theta_j^{(r)} & 0 & \dots & 0 & \dots & \phi_{k_r}^{(r)} \end{vmatrix} H, \qquad \dots(2.10)$$

$$1 \leqslant j \leqslant \lambda, \quad 1 \leqslant k_i \leqslant \nu^{(i)}, \quad i = 1, \dots, r.$$

This includes as a special case (3.4) of Tandon (1980) for the choice $j=k_1=\ldots=k_r=1$. (3.5) and (3.6) of Tandon (1980) would also follow from (2.10) by the following replacements:

$$1-a_j \to a_j-1;\ \theta_{k_i}^{(i)} \to -\theta_{k_i}^{(i)};\ j=A,\ k_1=\ldots=k_r=1$$

and

$$H[a_j-1] \to H[c_j+1];\quad 1-a_j \to -c_j;\ \theta_{k_i}^{(i)} \to \psi_{k_j}^{(i)};$$

$$j=k_1=\ldots k_r=1,\quad i=1,\ldots,r,$$

respectively. These results, in view of the cases of reducibility of the H-function (Saxena 1977) would generate contiguous relations for Lauricella's r-ple function. For one such typical relation for $F_A^{(r)}$ one may refer to [Tandon 1980, (5.2)]. Also they include the 4-term contiguous relations as obtained in Buschman and Gupta (1975) for the two variable case of the H-function. Note here that Tandon (1980) obtains his (3.4) – (3.6) and the three term contiguous relations whose mention will be made later, by a different technique.

On the other hand, a $(r+2)$-term contiguous function relation with $(r+2)$ contiguous functions, which can either be obtained by the process of elimination between two contiguous function relations with $(r+1)$ contiguous functions or by the foregoing analysis, can be exhibited in the following form:

$$\begin{vmatrix} H[a_1-1] & \ldots & H[a_k-1] & H[c_1+1] & \ldots & H[c_p+1] \\ 1-a_1 & \ldots & 1-a_k & -c_1 & \ldots & -c_p \\ \theta_1' & \ldots & \theta_k' & \psi_1' & \ldots & \psi_p' \\ . & \ldots & . & . & \ldots & . \\ \theta_1^{(i)} & \ldots & \theta_k^{(i)} & \psi_1^{(i)} & \ldots & \psi_p^{(i)} \\ . & \ldots & . & . & \ldots & . \\ \theta_1^{(r)} & \ldots & \theta_k^{(r)} & \psi_1^{(r)} & \ldots & \psi_p^{(r)} \end{vmatrix}$$

$$\begin{vmatrix} H[b_1^{(i)}-1] & \ldots & H[b_m^{(i)}-1] & H[d_1^{(i)}+1] & \ldots & H[d_{r+2-k-p-m}^{(i)}+1 \\ 1-b_1^{(i)} & \ldots & 1-b_m^{(i)} & d_1^{(i)} & \ldots & d_{r+2-k-p-m}^{(i)} \\ 0 & \ldots & 0 & 0 & \ldots & 0 \\ . & \ldots & . & . & \ldots & . \\ \phi_1^{(i)} & \ldots & \phi_m^{(i)} & -\delta_1^{(i)} & \ldots & -\delta_{r+2-k-p-m}^{(i)} \\ . & \ldots & . & . & \ldots & . \\ 0 & \ldots & 0 & 0 & \ldots & 0 \end{vmatrix} = 0, \quad \ldots(2.11)$$

$$1 \leqslant k \leqslant \lambda,\ 1 \leqslant p \leqslant C,\ 1 \leqslant m \leqslant \nu^{(i)},\ 1 \leqslant r+2-k-p-m \leqslant \mu^{(i)},\ i=1,\ldots,r,$$

We now ask a question, does there exist any contiguous relation whithout any first type of parameter. The answer is that whithout a first type parameter, there cannot exist any $(r+2)$ or less term contiguous relation which involves 'different kind' of second type parameters. On the other hand, three or more term contiguous function relations, involving only same kind of second type parameters, can exist.

Indeed, from the foregoing analysis, all three-term contiguous function relations involving two contiguous functions relative to the 'same kind' of 'second type' parameters can be written in 10 sets of relations. Using the determinant notation, which displays an inherent symmetry, in preference to that of Buschman's (1972) notation, three sets of relations that are basic and generate the remaining seven are given below:

$$\text{(i)} \quad \begin{vmatrix} H[b_j^{(i)}-1] & H[d_k^{(i)}+1] \\ -\phi_j^{(i)} & -\delta_k^{(i)} \end{vmatrix} = \begin{vmatrix} b_j^{(i)}-1 & d_k^{(i)} \\ -\phi_j^{(i)} & -\delta_k^{(i)} \end{vmatrix} H,$$

$$\nu^{(i)}+1 \leqslant j \leqslant B^{(i)}, \quad 1 \leqslant k \leqslant \mu^{(i)};$$

$$\text{(ii)} \quad \begin{vmatrix} H[b_j^{(i)}-1] & H[d_k^{(i)}+1] \\ \phi_j^{(i)} & \delta_k^{(i)} \end{vmatrix} = \begin{vmatrix} 1-b_j^{(i)} & -d_k^{(i)} \\ \phi_j^{(i)} & \delta_k^{(i)} \end{vmatrix} H,$$

$$1 \leqslant j \leqslant \nu^{(i)}, \quad \mu^{(i)}+1 \leqslant k \leqslant D^{(i)};$$

$$\text{(iii)} \quad \begin{vmatrix} H[b_j^{(i)}-1] & H[b_k^{(i)}-1] \\ \phi_j^{(i)} & -\phi_k^{(i)} \end{vmatrix} = \begin{vmatrix} 1-b_j^{(i)} & b_k^{(i)}-1 \\ \phi_j^{(i)} & -\phi_k^{(i)} \end{vmatrix} H,$$

$$1 \leqslant j \leqslant \nu^{(i)}, \quad \nu^{(i)}+1 \leqslant k \leqslant B^{(i)}, \quad i = 1, \ldots, r. \qquad \ldots(2.12)$$

However, with the restrictions

$$\left.\begin{aligned} \theta_j' = \theta_j^{(2)} = \ldots = \theta_j^{(r)} = \theta_j; \quad j = 1, \ldots, A \\ \psi_j' = \psi_j^{(2)} = \ldots = \psi_j^{(r)} = \psi_j; \quad j = 1, \ldots, C \end{aligned}\right\} \qquad \ldots(2.13)$$

of a total of six sets of three-term contiguous relations involving only two first type parameters, the two sets that are basic are as follows:

$$\text{(i)} \quad \begin{vmatrix} H[a_j-1] & H[a_k-1] \\ \theta_j & -\theta_k \end{vmatrix} = \begin{vmatrix} 1-a_j & a_k-1 \\ \theta_j & -\theta_k \end{vmatrix} H,$$

$$1 \leqslant j \leqslant \lambda, \quad \lambda+1 \leqslant k \leqslant A;$$

(ii) $$\begin{vmatrix} H[a_j - 1] & H[c_k + 1] \\ -\theta_j & \psi_k \end{vmatrix} = \begin{vmatrix} a_j - 1 & -c_k \\ -\theta_j & \psi_k \end{vmatrix} H,$$

$$\lambda + 1 \leqslant j \leqslant A,\ 1 \leqslant k \leqslant C. \qquad \ldots(2.14)$$

Whereas the identities (2.12) include Tandon's [1980, (3.1) – (3.3)] the identities governed by the sets (2.14) do not seem to have been noticed earlier.

An analysis similar to that of (2.11), now enables us to write a three-term contiguous relation with three contiguous functions in the form:

$$\begin{vmatrix} H[b_j^{(i)} - 1] & H[b_k^{(i)} - 1] & H[d_m^{(i)} + 1] \\ 1 - b_j^{(i)} & b_k^{(i)} - 1 & d_m^{(i)} \\ \phi_j^{(i)} & -\phi_k^{(i)} & -\delta_m^{(i)} \end{vmatrix} = 0, \qquad \ldots(2.15)$$

$$i \leqslant j \leqslant \nu^{(i)},\ \nu^{(i)} + 1 \leqslant k \leqslant B^{(i)},\ 1 \leqslant m \leqslant \mu^{(i)},\ i = 1, \ldots, r.$$

It is of interest to note that there are twenty sets of such relations for 'same kind' of 'second type' parameters and ten sets for 'first type' parameters with the restrictions as in (2.13). Various combinations of contiguous relations listed in (2.12) or (2.14) for different choices of indices involved will generate four-term, five-term etc. contiguous relations. For reasons of brevity, we omit the details.

In summing up the foregoing discussion, it would be worthwhile to give comprehensive schemes for writing $(r + 2)$-term contiguous relations and three-term contiguous relations.

A. *Scheme for* $(r + 2)$*-term Contiguous Relation Involving* $(r + 1)$ *Parameter with atleast One 'First Type' Parameter*

(i) The relation is of the form $X = YH$, where both X and Y are determinants of order $(r + 1)$ and H is the H-function as in defining integral (1.1).

(ii) For the given set of $(r + 1)$ parameters for which the contiguous relation has to be obtained, for each parameter one has to look the corresponding multiplying factor as envisaged by the list (2.2) – (2.5).

(iii) Express the multiplying factor corresponding to a parameter as $u_0 + u_1 s_1 + \ldots + u_r s_r$. The column matrix $[u_0, u_1, \ldots, u_r]$ which is displayed here by a row, will form the column of the determinant Y corresponding to that parameter.

(iv) Having thus constructed Y, to construct X, one has to replace elements of the first row of Y by the contiguous functions of their representative parameters keeping in view the list (2.2) – (2.5). The remaining rows of X should be reproduced as such from Y.

For illustrations, one may refer ro (2.8) – (2.10).

B. *Scheme for* $(r + 2)$*-term Contiguous Relation Involving* $(r + 2)$ *Parameters with atleast One First Type Parameter*

(i) The contiguous relation has the form $Z = 0$, where Z is a determinant of order $(r + 2)$.

(ii) Corresponding to each parameter, one has to look for the corresponding contiguous function and multiplying factor as envisaged by the list (2.2) – (2.5).

(iii) Express the multiplying factor as

$$u_0 + u_1 s_1 + \ldots + u_r s_r.$$

The column matrix

[contiguous function, u_0, u_1, ..., u_r],

displayed here as row, will form a column of Z corresponding to that parameter. The construction of Z is thus complete.

For illustration, reference may be made to (2.11).

Analogous schemes may be described for writing three-term contiguous relations involving either two contiguous functions or three contiguous functions.

3. Finite Summations

The three-term contiguous relations give rise to interesting finite summation formulas by formation of collapsing series. As an illustration, consider (2.14 ii) where it is being assumed that $\lambda + 1 \leqslant j \leqslant A$, $0 < k \leqslant C$.

If we pair the terms which involve a_j and write

$$\psi_k H[a_j - 1, c_k + m - 1]$$

$$= -\theta_j H[a_j, c_k + m] + \begin{vmatrix} a_{j-1} & -(c_k + m - 1) \\ -\theta_j & \psi_k \end{vmatrix} H[a_j, c_k + m - 1] \qquad \ldots(3.1)$$

then expressing the determinant in (3.1) in the form

$$\begin{vmatrix} a_j - 1 & -(c_k + m - 1) \\ -\theta_j & \psi_k \end{vmatrix} = \frac{-\theta_j \Gamma\left[c_k + m - (a_j - 1)\frac{\psi_k}{\theta_j}\right]}{\Gamma\left[c_k + m - 1 - (a_j - 1)\frac{\psi_k}{\theta_j}\right]} \qquad \ldots(3.2)$$

summing on m and collapsing the resulting series on the right, we have the finite summation formula:

$$\frac{\psi_k}{\theta_j}\sum_{m=1}^{n}\frac{(-1)^m H[a_j-1,\ c_k+m-1]}{\Gamma[c_k+m-(a_j-1)\psi_k\theta_j^{-1}]}=\frac{(-1)^{n+1}H[a_j,\ c_k+n]}{\Gamma[C_k+n-(a_j-1)\psi_k\theta_j^{-1}]}+\frac{H[a_j,\ c_k]}{\Gamma[c_k-(a_j-1)\psi_k\theta_j^{-1}]}. \quad \ldots(3.3)$$

Note also that (3.2) admits the alternative form:

$$\theta_j\Gamma\left[(a_j-1)\frac{\psi_k}{\theta_j}-(c_k+m-2)\right]\Big/\Gamma\left[(a_j-1)\frac{\psi_k}{\theta_j}-(c_k+m-1)\right].$$

Consequently from (3.1) we have a similar formula:

$$\frac{\psi_k}{\theta_j}\sum_{m=1}^{n}\Gamma\left[(a_j-1)\frac{\psi_k}{\theta_j}-(c_k+m-1)\right]H[a_j-1,\ c_k+m-1]$$
$$=\Gamma\left[(a_j-1)\frac{\psi_k}{\theta_j}-(c_k-1)\right]H[a_j,\ c_k]$$
$$-\Gamma\left[(a_j-1)\frac{\psi_k}{\theta_j}-(c_k+n-1)\right]H[a_j,\ c_k+n]. \quad \ldots(3.4)$$

A different pairing of terms in (2.14 ii) with the same assumption on indices as before, leads us to the following additional finite summation formulas:

$$\frac{\theta_j}{\psi_k}\sum_{m=1}^{n}\frac{(-1)^m H[a_j-m+1,\ c_k+1]}{\Gamma[c_k\theta_j\psi_k^{-1}-a_j+m+1]}=\frac{(-1)^n H[a_j-n,\ c_k]}{\Gamma[c_k\theta_j\psi_k^{-1}-a_j+n+1]}+\frac{H[a_j,\ c_k]}{\Gamma[c_k\theta_j\psi_k^{-1}-a_j+1]} \quad \ldots(3.5)$$

$$\frac{\theta_j}{\psi_k}\sum_{m=1}^{n}\Gamma\left[a_j-m-c_k\frac{\theta_j}{\psi_k}\right]H[a_j-m+1,\ c_k+1]$$
$$=\Gamma\left[a_j-c_k\frac{\theta_j}{\psi_k}\right]H[a_j,\ c_k]-\Gamma\left[a_j-n-c_k\frac{\theta_j}{\psi_k}\right]H[a_j-n,\ c_k] \quad \ldots(3.6)$$

$$\sum_{m=1}^{n}(-1)^m\theta_j^{m-1}\psi_k^{n-m}\begin{vmatrix} a_j+m-1 & -(c_k+m) \\ -\theta_j & \psi_k \end{vmatrix}H[a_j+m,\ c_k+m]$$
$$=(-1)^n\theta_j^n H[a_j+n,\ c_k+n+1]-\psi_k^n H[a_j,\ c_k+1]. \quad \ldots(3.7)$$

Finite summation formulas corresponding to remaining three-term contiguous relations in (2.12) and (2.14) and those generated by them can be obtained similarly.

Relations (3.3) – (3.7) in particular will yield finite summation formulas involving Lauricella's r-ple hypergeometric functions. In particular from (3.5), we have

$$\frac{(c-1)\,\Gamma(c)}{\Gamma(c-1)} \sum_{m=1}^{p} \frac{(-1)^m\,\Gamma(a+m-1)}{\Gamma(a-c+m+1)} \times F_D^{(n)}(a+m-1, b_1, \ldots, b_n, c-1; x_1, \ldots, x_n)$$

$$= (-1)^p \frac{\Gamma(a+p)}{\Gamma(a-c+p+1)} F_D^{(n)}(a+p; b_1, \ldots, b_n; c; x_1, \ldots, x_n) + \Gamma(a)\, F_D^{(n)}(a; b_1, \ldots, b_n; c; x_1, \ldots, x_n). \quad \ldots(3.8)$$

Further reducibility of this in terms of Appell's function F_1 is rather straightforward and we have

$$\frac{(c-1)\,\Gamma(c)}{\Gamma(c-1)} \sum_{m=1}^{p} \frac{(-1)^m\,\Gamma(a+m-1)}{\Gamma(a-c+m+1)} F_1(a+m-1; b_1, b_2; c-1; x_1, x_2)$$

$$= (-1)^p \frac{\Gamma(a+b)}{\Gamma(a-c+p+1)} F_1(a+p; b_1, b_2; c; x_1, x_2) + \Gamma(a)\, F_1(a; b_1, b_2; c; x_1, x_2). \quad \ldots(3.9)$$

4. On Derivatives and Partial Derivatives

In this section, methods involving derivatives and the Mellin transformation are employed in obtaining finite summations for the H-function and certain special partial derivatives for the H-function with respect to parameters.

If t be an arbitrary parameter and $\alpha', \ldots, \alpha^{(r)}$ be positive real numbers, then it can be verified that

$$D^n \left| t^{e-1} H\left(\frac{z_1}{t^{\alpha'}}, \ldots, \frac{z_r}{t^{\alpha^{(r)}}}\right)\right.$$

$$= (-1)^n\, t^{e-n-1} H^{0,\lambda+1:(\mu',\nu');\ldots;(\mu^{(r)},\nu^{(r)})}_{A+1,C+1:[B',D'];\ldots;[B^{(r)},D^{(r)}]} \left(\begin{matrix} [e-n:\alpha', \ldots, \alpha^{(r)}], [(a); \theta', \ldots, \theta^{(r)}] : [(b'):\phi']; \ldots; \\ [e:\alpha', \ldots, \alpha^{(r)}], [(c):\psi', \ldots, \psi^{(r)}] : [(d'):\delta'], \ldots, \\ [(b^{(r)}):\phi^{(r)}]; \\ [(d^{(r)}); \delta^{(r)}]; \end{matrix} \frac{z_1}{t^{\alpha'}}, \ldots, \frac{z_r}{t^{\alpha^{(r)}}}\right) \quad \ldots(4.1)$$

and

$$t^{\sigma-1} H^{0,0:(\mu',\nu');\ldots;(\mu^{(r)},\nu^{(r)})}_{0,0:[B',D'];\cdots;[B^{(r)},D^{(r)}]} \left(\begin{matrix} -:[(b'):\phi']; \ldots; \\ -:[(d'):\delta']; \ldots; \end{matrix} \right.$$

$$\left. \begin{matrix} [(b^{(r)}):\phi^{(r)}]; \\ [(d^{(r)}):\delta^{(r)}]; \end{matrix} \quad \frac{z_1}{t^{\alpha'}}, \ldots, \frac{z_r}{t^{\alpha^{(r)}}} \right)$$

$$= t^{(\sigma-1)/r} H^{\mu',\nu'}_{B',D'} \left(\begin{matrix} [(b'):\phi'] \\ [(d'):\delta'] \end{matrix} ; \frac{z_1}{t^{\alpha'}} \right), \ldots, t^{(\sigma-1)/r}$$

$$\times H^{\mu^{(r)},\nu^{(r)}}_{B^{(r)},D^{(r)}} \left(\begin{matrix} [(b^{(r)}):\phi^{(r)}] \\ [(d^{(r)}):\delta^{(r)}] \end{matrix} ; \frac{z_r}{t^{\alpha^{(r)}}} \right). \qquad \ldots(4.2)$$

Differentiating (4.2) n times w.r.t. t and simplifying, it follows by induction that

$$H^{0,1:(\mu',\nu');\ldots;(\mu^{(r)},\nu^{(r)})}_{1,1:[B',D'];\cdots;[B^{(r)},D^{(r)}]} \left(\begin{matrix} [e-n:\alpha', \ldots, \alpha^{(r)}]:[(b'):\phi']; \ldots; [(b^{(r)}):\phi^{(r)}]; \\ [e:\alpha', \ldots, \alpha^{(r)}]:[(d'):\delta']; \ldots; [(d^{(r)}):\delta^{(r)}]; \end{matrix} \right.$$

$$\left. ; \frac{z_1}{t^{\alpha'}}, \ldots, \frac{z_r}{t^{\alpha^{(r)}}} \right)$$

$$= \sum_{\substack{n_1,\ldots,n_r=0 \\ n_1+\ldots+n_r=n}}^{n} \frac{n!}{n_1! \ldots n_r!} H^{0,0:(\mu',\nu'+1);\ldots;(\mu^{(r)},\nu^{(r)}+1)}_{0,0:[B'+1,D'+1];\cdots;[B^{(r)}+1,D^{(r)}+1]}$$

$$\left(\begin{matrix} -:\left[\frac{e+r-1}{r} - n_1 : \alpha'\right], [(b'):\phi']; -; \\ -:[(d'):\delta'], \left[\frac{e+r-1}{r} : \alpha'\right]; -; \end{matrix} \right.$$

$$\left. \begin{matrix} \left[\frac{e+r-1}{r} - n_r : \alpha^{(r)}\right], [(b^{(r)}):\phi^{(r)}]; \\ [(d^{(r)}):\delta^{(r)}], \left[\frac{e+r-1}{r} : \alpha^{(r)}\right] \end{matrix} \quad , \frac{z_1}{t^{\alpha'}}, \ldots, \frac{z_r}{t^{\alpha^{(r)}}} \right). \qquad \ldots(4.3)$$

(4.3) readily admits an extension and we have

$$H^{0,\lambda+1:(\mu',\nu');\ldots;(\mu^{(r)},\nu^{(r)})}_{A+1,C+1:[B',D'];\cdots;[B^{(r)},D^{(r)}]} \left(\begin{matrix} [e-n:\alpha', \ldots, \alpha^{(r)}], [(a):\theta', \ldots, \theta^{(r)}]: \\ [e:\alpha', \ldots, \alpha^{(r)}], [(c):\psi', \ldots, \psi^{(r)}]: \end{matrix} \right.$$

$$\left. \begin{matrix} [(b'):\phi']; \ldots; [(b^{(r)}):\phi^{(r)}]; \\ [(d'):\delta']; \ldots; (d^{(r)}):\delta^{(r)}]; \end{matrix} \quad \frac{z_1}{t^{\alpha'}}, \ldots, \frac{z_r}{t^{\alpha^{(l)}}} \right) =$$

(equation continued on p. 838)

$$= \sum_{\substack{n_1,\ldots,n_r=0 \\ n_1+\ldots+n_r=n}}^{n} \frac{n!}{n_1!\ \ldots\ n_r!}\, H^{0,\lambda:(\mu',\nu'+1);\ldots;(\mu^{(r)},\nu^{(r)}+1)}_{A,C:[B'+1,D'+1];\ldots;[B^{(r)}+1,D^{(r)}+1]}$$

$$\left(\begin{matrix} [(a):\theta', \ldots, \theta^{(r)}] : \left[\frac{e+r-1}{r} - n_1 : \alpha'\right], [(b'):\phi']; \ldots; \\ [(c):\psi', \ldots, \psi^{(r)}] : [(d'):\delta'], \left[\frac{e+r-1}{r} : \alpha'\right]; \ldots; \\ \left[\frac{e+r-1}{r} - n_r : \alpha^{(r)}\right], [(b^{(r)}):\phi^{(r)}]; \\ [(d^{(r)}):\delta^{(r)}], \left[\frac{e+r-1}{r} : \alpha^{(r)}\right]; \end{matrix} \frac{z_1}{t^{\alpha'}}, \ldots, \frac{z_r}{t^{\alpha^{(t)}}} \right). \quad \ldots(4.4)$$

Considering various other forms that (4.1) admits, similar other results can be obtained.

In the next place, in view of (2.1) we note that the r-dimensional Mellin-transformation (Doetsch 1950; 11.2) $M^{(r)}$ of the H-function is given by

$$M^{(r)}(H) = Q(-s_1, \ldots, s_r)$$

provided

$$-\min \operatorname{Re}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) < s_i < \max \operatorname{Re}\left(\frac{1-b_j^{(i)}}{\phi_j^{(i)}}\right); \quad i = 1, \ldots, r.$$

$$1 \leqslant j \leqslant \mu^{(i)} \qquad 1 \leqslant j \leqslant \nu^{(i)}$$

We also note that, since [1.7 (30) of Erdelyi (1953)] for a positive integer N,

$$\psi(a+N) - \psi(a) = \sum_{k=1}^{N} \frac{(-1)^{k-1}\, N!}{k(N-K)!}\, \frac{\psi(a)}{\psi(a+K)}, \quad \psi(a) = \frac{\Gamma'(a)}{\Gamma(a)},$$

partial differentiation of the gamma product $\Gamma(\frac{1}{2}e + \sum_{i=1}^{r} \alpha^{(i)}s_i)\, \Gamma(-\frac{1}{2}e + \sum_{i=1}^{r} \alpha^{(i)}s_i)$ w.r.t. the arbitrary parameter e at $e = N$ can be expressed as a finite sum

$$\frac{\partial}{\partial e}\left\{\Gamma\left(\tfrac{1}{2}e + \sum_{i=1}^{r} \alpha^{(i)}s_i\right)\Gamma\left(-\tfrac{1}{2}e + \sum_{i=1}^{r} \alpha^{(i)}s_i\right)\right\}\Big|_{e=N}$$

$$= \tfrac{1}{2}\Gamma(\tfrac{1}{2}N + \sum_{i=1}^{r} \alpha^{(i)}s_i) \times$$

(equation continued on p. 839)

$$\times \sum_{K=1}^{N} \frac{N!\,(-1)^{K-1}}{K(N-K)!} \frac{\Gamma(-\frac{1}{2}N + \sum_{i=1}^{r} \alpha^{(i)} s_i)}{\Gamma(-\frac{1}{2}N + K + \sum_{i=1}^{r} \alpha^{(i)} s_i)},$$

where $\alpha', \ldots, \alpha^{(r)}$ are positive real members.

Thus for $\lambda > 0$, $N > 0$, we have

$$M^{(r)} \left\{ \frac{\partial}{\partial e} \left[H^{0,\lambda+2:(\mu',\nu');\ldots;(\mu^{(r)},\nu^{(r)})}_{A+2,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]} \left(\begin{matrix} [1+\frac{1}{2}e:\alpha',\ldots,\alpha^{(r)}], [1-\frac{1}{2}e:\alpha',\ldots,\alpha^{(r)}], \\ [(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(c):\psi',\ldots,\psi^{(r)}]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \end{matrix} z_1, \ldots, z_r \right) \right]_{e=N} \right\} =$$

$$= \frac{N!}{2} \sum_{K=1}^{N} \frac{(-1)^{k-1}\Gamma(\frac{1}{2}N - \sum_{i=1}^{r} \alpha^{(i)} s_i)\,(\Gamma(-\frac{1}{2}N - \sum_{i=1}^{r} \alpha^{(i)} s_i))^2}{K(N-K)!\,\Gamma(-\frac{1}{2}N + K - \sum_{i-1}^{r} \alpha^{(i)} s_i)}$$

$$\times Q(-s_1, \ldots, -s_r)$$

$$= M^{(r)} \left\{ \frac{N!}{2} \sum_{K=1}^{N} \frac{(-1)^{K-1}}{K(N-k)!} H^{0,\lambda+3:(\mu',\nu');\ldots;(\mu^{(r)},\nu^{(r)})}_{A+3,C+1:[B',D'];\cdots;[B^{(r)},D^{(r)}]} \right.$$

$$\left(\begin{matrix} [1+\frac{1}{2}N:\alpha',\ldots,\alpha^{(r)}], [1+\frac{1}{2}N:\alpha',\ldots,\alpha^{(r)}], \\ [1-\frac{1}{2}N:\alpha',\ldots,\alpha^{(r)}], [(a):\theta',\ldots,\theta^{(r)}]:[(b'):\psi'];\ldots; \\ [1+\frac{1}{2}N-K:\alpha',\ldots,\alpha^{(r)}], [(c):\psi',\ldots,\psi^{(r)}]:[(d'):\delta'];\ldots; \\ [(b^{(r)}):\psi^{(r)}]; \\ [(d^{(r)}):\delta^{(r)}]; \end{matrix} z_1, \ldots, z_r \right) \Bigg\}. \quad \ldots(4.5)$$

But for $z_i = u^{-\alpha^{(i)}}$, (4.5) can be written again as

$$\frac{\partial}{\partial e} H^{0,\lambda+2:(\mu',\nu');\ldots;(\mu^{(r)},\nu^{(r)})}_{A+2,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]} \left(\begin{matrix} [1+\frac{1}{2}e:\alpha',\ldots,\alpha^{(r)}], [1-\frac{1}{2}e:\alpha',\ldots,\alpha^{(r)}], \\ [(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'];\ldots;[(b^{(r)}):\psi^{(r)}]; \\ [(c):\psi',\ldots,\psi^{(r)}]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; \end{matrix} z_1, \ldots, z_r \right) \Bigg|_{e=N} =$$

(equation continued on p. 840)

$$= \frac{N!}{2} \sum_{k=1}^{N} \frac{(-1)^N u^{N/2}}{(N-k)!\, k} D_u^{N-k} \left\{ u^{(N/2)-k} H^{0,\lambda+2:(\mu',\nu');\ldots;(\mu^{(r)},\nu^{(r)})}_{A+2,C:[A',B'];\cdots;[A^{(r)},B^{(r)}]} \right.$$

$$\left(\begin{matrix} [1 + \tfrac{1}{2}N : \alpha', \ldots, \alpha^{(r)}], [1 - \tfrac{1}{2}N : \alpha', \ldots, \alpha^{(r)}], [(a) : \theta', \ldots, \theta^{(r)}] : \\ [(c) : \psi', \ldots, \psi^{(r)}] : \\ [(b') : \phi']; \ldots; [(b^{(r)}) : \phi^{(r)}]; \\ [(d') : \delta']; \ldots; [(d^{(r)}) : \delta^{(r)}]; \end{matrix} z_1, \ldots, z_r \right) \Bigg\}. \qquad \ldots(4.6)$$

If we expand the derivative into a sum, carry out the differentiations, interchange the order of summation and simplify, we obtain

$$\frac{\partial}{\partial e} H^{0,\lambda+2:(\mu',\nu');\ldots;(\mu^{(r)},\nu^{(r)})}_{A+2,C:[B',D'];\cdots;[B^{(r)},D^{(r)}]} \left(\begin{matrix} [1 + \tfrac{1}{2}e : \alpha', \ldots, \alpha^{(r)}], [1 - \tfrac{1}{2}e : \alpha', \ldots, \alpha^{(r)}], \\ [(a) : \theta', \ldots, \theta^{(r)}]; [(b') : \phi']; \ldots; [(b^{(r)}) : \phi^{(r)}]; \\ [(C) : \psi', \ldots, \psi^{(r)}]; [(d') : \delta']; \ldots; [(d^{(r)}) : \delta^{(r)}]; \end{matrix} z_1, \ldots, z_r \right) \Bigg|_{e=N}$$

$$= \frac{N!}{2} \sum_{p=0}^{N-1} \frac{1}{(N-p)\, p!} H^{0,\lambda+2:(\mu',\nu');\ldots;(\mu^{(r)},\nu^{(r)})}_{A+2:C:[B',D'];\cdots;[B^{(r)},D^{(r)}]}$$

$$\left(\begin{matrix} [1 + \tfrac{1}{2}N; \alpha', \ldots, \alpha^{(r)}], [1 + \tfrac{1}{2}N - p : \alpha', \ldots, \alpha^{(r)}], [(a) : \theta', \ldots, \theta^{(r)}] : \\ [(c) : \psi', \ldots, \psi^{(r)}] : \\ [(b') : \phi']; \ldots; [(b^{(r)}) : \phi^{(r)}]: \\ [(d') : \delta']; \ldots; [(d^{(r)}) : \delta^{(r)}]; \end{matrix} z_1, \ldots, z_r \right). \qquad \ldots(4.7)$$

Similar other results can be obtained by considering products or quotients of such gamma functions whose partial derivatives w.r.t. the arbitrary parameter involved can be expressed as a finite sum.

For example, for the quotient

$$\Gamma(e + N + \sum_{i=1}^{r} \alpha^{(i)} s_i) / \Gamma(e + \sum_{i=1}^{r} \alpha^{(i)} s_i),$$

we have

$$\frac{\partial}{\partial e} H^{0,\lambda+1:(\mu',\nu');\ldots;(\mu^{(r)},\nu^{(r)})}_{A+1,C+1:[B',D'];\cdots;[B^{(r)},D^{(r)}]} \left(\begin{matrix} [1 - e - N : \alpha', \ldots, \alpha^{(r)}], [(a); \theta', \ldots, \theta^{(r)}]: \\ [1 - e : \alpha', \ldots, \alpha^{(r)}], [(c); \psi', \ldots, \psi^{(r)}] : \\ [(b') : \phi']; \ldots; [(b^{(r)}) : \phi^{(r)}]; \\ [(d') : \delta']; \ldots; [(d^{(r)}) : \delta^{(r)}]; \end{matrix} z_1, \ldots, z_r \right) =$$

(*equation continued on p.* 841)

$$= N! \sum_{k=1}^{N} \frac{(-1)^{k-1}}{k!(N-k)!} H^{0,\lambda+1:(\mu',\nu');\ldots;(\mu^{(r)},\nu^{(r)})}_{A+1,C+1:[B',D'];\ldots;[B^{(r)},D^{(r)}]}$$

$$\left(\begin{matrix} [1-e-N:\alpha',\ldots,\alpha^{(r)}],[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'];\ldots; \\ [1-e-k:\alpha',\ldots,\alpha^{(r)}],[(c):\psi',\ldots,\psi^{(r)}]:[(d'):\delta'];\ldots; \end{matrix} \begin{matrix} [(b^{(r)}):\phi^{(r)}]; \\ [(d^{(r)}):\delta^{(r)}]; \end{matrix} z_1,\ldots,z_r \right). \quad \ldots(4.8)$$

One may also obtain multiple r-analogues of Deshpande type [1979, (2.1) – (2.4)] finite summation formulas involving 'second type' of parameters. Because of the distinct nature of the 'first type' and the 'second type' parameters, these results will be different from our results (4.7) and (4.8). It may further be remarked in passing that formulas (4.3) – (4.4), and (4.7) – (4.8) respectively provide extensions of Deshpande's (1971), and Buschman's (1974) results.

5. Characterization as Probability Density Function

Let $x_1, \ldots, x_p$ be p independent random variables which are distributed according to the probability law

$$f(x) = \begin{cases} \dfrac{\beta a^{\alpha/\beta}}{\alpha/\beta} x^{\alpha-1} e^{-ax^{\beta}}; & x > 0, \alpha > 0, \beta > 0, a > 0 \\ 0 & ; \text{elsewhere,} \end{cases} \quad \ldots(5.1)$$

Let

$$Y_i = \frac{X_{n_{i-1}-1} \ldots X_{m_i}}{X_{m_i+1} \ldots X_{n_i}}, \quad i = 1, \ldots, r,$$

$$n_0 = 0,\ n_r = p,\ n_{i-1} < m_i < n_i,\ 2r \leqslant p\ ,$$

and

$$Y = Y_1 \ldots Y_r.$$

If the densities $g(y_i)$ of Y_i and $g(y)$ of Y exist, then, since $Y_1, \ldots, Y_i$ are independent,

$$g(y) = g(y_1) \ldots g(y_r).$$

But, the $(s-1)$th moment of Y_i about the origin is given by

$$E(Y_i^{s-1}) = \prod_{j=n_{i-1}+1}^{m_i} \{E(X_j^{(s-1)})\} \prod_{j=m_i+1}^{n_i} \{E(X_j^{-(s-1)})\}$$

$$= \frac{a^{(2m_i-(n_i-n_{i-1}))/\beta} \cdot \Gamma^{(m_i-n_{i-1})}((\alpha-1)/\beta + (s/\beta))\,\Gamma^{(n_i-m_i)}((\alpha+1)/\beta - (s/\beta))}{a^{(2m_i-(n_i-n_{i-1}))s/\beta}\,\Gamma^{(n_i-n_{i-1})}(\alpha/\beta)} \quad \ldots(5.2)$$

where E denotes the mathematical expectation.

Therefore, the density function $g(y_i)$ of Y_i can be extracted from (5.2) by the inverse Mellin-transformation, that is:

$$g(y_i) = \frac{a^{(2m_i-(n_i-n_{i-1}))/\beta}}{\Gamma^{(n_i-n_{i-1})}(\alpha/\beta)} H^{m_i-n_{i-1},n_i-m_i}_{n_i-m_i,m_i-n_{i-1}} \times \left(\begin{matrix} \left[\left(1-\frac{\alpha+1}{\beta}:\frac{1}{\beta}\right):n_i-m_i\right]; \\ \left[\left(\frac{\alpha-1}{\beta}:\frac{1}{\beta}\right);m_i-n_{i-1}\right]; \end{matrix} a^{(2m_i-(n_i-n_{i-1}))/\beta}\cdot y_i \right), \quad \ldots(5.3)$$

where $[(a:b);n]$ denotes the sequence of n parameters each equal to $(a:b)$. Consequently, we have

$$g(y) = \frac{a^*}{\Gamma^p(\alpha/\beta)} H^{0,0:(m_1-n_0,n_1-m_1);\ldots;(m_r-n_{r-1},n_r-m_r)}_{0,0:[n_1-m_1,m_1-n_0];\cdots;[n_r-m_r,m_r-n_{r-1}]} \left(\begin{matrix} -:\left[\left(1-\frac{\alpha+1}{\beta}:\frac{1}{\beta}\right);n_1-m_1\right];\ldots;\left[\left(1-\frac{\alpha+1}{\beta}:\frac{1}{\beta}\right);\right. \\ -:\left[\left(\frac{\alpha-1}{\beta}:\frac{1}{\beta}\right);m_1-n_0\right];\ldots;\left[\left(\frac{\alpha-1}{\beta}:\frac{1}{\beta}\right);\right. \end{matrix} \begin{matrix} n_r-m_r\Big]; \\ m_r-n_{r-1}\Big]; \end{matrix} a^{(2m_1-(n_1-n_0))/\beta}\cdot y_1,\ldots,a^{(2m_r-(n_r-n_{r-1}))/\beta}\cdot y_r \right),$$

$$* = 2\sum_{i=1}^{r} m_i - p. \quad \ldots(5.4)$$

Further extensions of (5.4) are possible by considering probability density functions of a more general nature than considered in (5.1). For a general discussion of such problems, one may refer to the work of Mathai (1972) and Srivastava and Singhal (1972).

It may be added in conclusion that so long as the ratios $Y_1, \ldots, Y_r$ are independent, whether we consider the probability law studied by us or the one studied by Srivastava and Singhal (1972), p.d.f. for the product of ratios would always be expressed as a product of r H-functions of one variable or for that matter, as a particular multiple H-function. In case when the ratios are dependent or the distribution is a multivariate one, it is conjectured that the p.d.f. would be expressible in terms of a general multiple H-function. We, however, leave the problem for a subsequent communication.

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