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MULTIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Let T_p be the class of regular and p-valent functions which can be expressed in the form

$$f(z) = z^p - \sum_{n=1}^{\infty} | a_{n+p} | z^{n+p}, \quad | z | < 1.$$

The subclasses $T_p^*(A, B)$ and $C_p(A, B)$ of T_p have been considered. Sharp results concerning coefficient estimates, distortion and covering theorems are obtained. The radius of convexity for the class $T_p^*(A, B)$ is determined. It is further proved that the classes $T_p^*(A, B)$ and $C_p(A, B)$ are closed under arithmetic mean and convex linear combinations.

1. Introduction

Let S_p denote the class of functions of the form $f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$ which are analytic and p-valent in the unit disc $E = \{z: | z | < 1\}$. A function f is said to be subordinate to a function $F (f < F)$ if there exists an analytic function $\phi(z)$ with $| \phi(z) | \leqslant | z |$, $z \in E$, such that $f = F \circ \phi$. For A, B fixed, $-1 \leqslant A < B \leqslant 1$, we say that $f \in S_p^* (A, B)$ if and only if $\frac{zf'(z)}{f(z)} < p \frac{1 + Az}{1 + Bz}$, $z \in E$, or equivalently $f \in S_p^* (A, B)$ if and only if

$$\left| \left(\frac{zf'(z)}{f(z)} - p \right) \Big/ \left(\frac{Bzf'(z)}{f(z)} - Ap \right) \right| < 1, \qquad z \in E.$$

Further f is said to belong to the class $K_p (A, B)$ if and only if

$$\frac{zf'}{p} \in S_p^* (A, B).$$

Let T_p denote the subclass of S_p consisting of functions regular and p-valent which can be expressed in the form

$$f(z) = z^p - \sum_{n=1}^{\infty} | a_{p+n} | z^{p+n}.$$

We denote by $T_p^*(A, B)$ and $C_p(A, B)$ the classes obtained by taking intersections respectively of the classes $S_p^*(A, B)$ and $K_p(A, B)$ with T_p.

Silverman (1975), Gupta and Jain (1976) and Silverman and Silvia (1978, 1979) have studied certain subclasses of univalent functions with negative coefficients. In this paper we obtain coefficient estimates, distortion and covering theorems for the classes $T_p^*(A, B)$ and $C_p(A,B)$. We also determine the radius of convexity for the class $T_p^*(A, B)$. It is further shown that the classes $T_p^*(A, B)$ and $C_p(A, B)$ are closed under arithmetic mean and convex linear combinations. By assigning specific values to A and B and taking $p = 1$, we get results due to Silverman (1975) and Gupta and Jain (1976).

2· Coefficient Inequalities

Theorem 1 — A function $f(z) = z^p - \sum_{n=1}^{\infty} | a_{p+n} | z^{p+n}$ belongs to $T^*(A, B)$ if and only if

$$\sum_{n=1}^{\infty} [(1 + B)n + (B - A)p] \; | a_{p+n} | \leqslant (B - A)p. \qquad ...(1)$$

The result is sharp.

Proof : Let $| z | = 1$. Then

$$| zf'(z) - pf(z) | - | Bzf'(z) - Apf(z) | = | \sum_{n=1}^{\infty} - n | a_{p+n} | z^{p+n} |$$

$$- | (B + A)pz^p - \sum_{n=1}^{\infty} nB + (B - A)p \; | a_{p+n} | z^{p+n} |$$

$$\leqslant \sum_{n=1}^{\infty} [(1 + B) n + (B - A)p] \; | a_{p+n} | - (B - A) p \leqslant 0.$$

Hence by the principle of maximum modulus $f(z) \in T_p^*(A, B)$.

Conversely, suppose that

$$\left| \left(\frac{zf'(z)}{f(z)} - p \right) \Big/ \left(\frac{Bzf'(z)}{f(z)} - Ap \right) \right|$$

$$= \left| \frac{\sum_{n=1}^{\infty} n \mid a_{p+n} \mid z^{p+n}}{(B-A)pz^p - \sum_{n=1}^{\infty} [nB + (B-A)p] \mid a_{p+n} \mid z^{p+n}} \right|, z \in E.$$

$$< 1$$

Since $| \operatorname{Re} z | \leqslant | z |$ for all z, we have

$$\operatorname{Re} \left\{ \frac{\sum_{n=1}^{\infty} n \mid a_{p+n} \mid z^{p+n}}{(B-A)pz^p - \sum_{n=1}^{\infty} [nB + (B-A)p] \mid a_{p+n} \mid z^{p+n}} \right\} < 1. \qquad \ldots(2)$$

Choose values of z on the real axis so that $\frac{zf'(z)}{f(z)}$ is real. Upon clearing the denominator in (2) and letting $z - 1$ through real values, we get

$$\sum_{n=1}^{\infty} n \mid a_{p+n} \mid \leqslant [(B-A)p - \sum_{n=1}^{\infty} [n\,B + (B-A)p] \mid a_{p+n} \mid]$$

which implies that

$$\sum_{n=1}^{\infty} [(1+B)n + (B-A)p] \mid a_{p+n} \mid \leqslant (B-A)\,p.$$

The function

$$f(z) = z^p - \sum_{n=1}^{\infty} \frac{(B-A)p}{(1+B)n + (B-A)p} z^{p+n}$$

is an extremal function.

Corollary 1 — If $f \in T_p^*(A, B)$ then $| a_{p+n} | \leqslant \frac{(B-A)p}{(1+B)n + (B-A)\,p}$, with equality only for functions of the from $f(z) = z^p - \frac{(B-A)\,p}{(1+B)\,n + (B-A)\,p} z^{p+n}$.

Corollary 2 — A function $f(z) = z^p - \sum_{n=1}^{\infty} | a_{p+n} | z^{p+n}$ belongs to $C_p(A, B)$ if and only if

$$\sum_{n=1}^{\infty} \left(\frac{n+p}{p} \right) [(1+B)n + (B-A)\,p] \mid a_{p+n} \mid \leqslant (B-A)p.$$

PROOF : $f \in C_p(A,B)$ if and only if $\frac{zf'}{p} \in T_p^*(A,B)$. Since

$$\frac{zf'}{p} = z^p - \sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right) | a_{p+n} | z^{p+n}$$

we may replace $| a_{p+n} |$ by $\left(\frac{n+p}{p}\right) | a_{p+n} |$ in the theorem.

Theorem 2 (Representation formula) — A function

$$f(z) = z^p - \sum_{n=1}^{\infty} | a_{n+p} | z^{n+p} \text{ is in } T_p^*(A,B) \text{ if and only if}$$

$$f(z) = z^p \exp\left(p(B-A) \int_0^z \frac{\phi(t)}{1 - Bt\,\phi(t)}\, dt \right), \qquad \ldots(3)$$

where $\phi(z)$ is analytic in E and satisfies $| \phi(z) | < 1$, $z \in E$.

PROOF : Let $f(z) \in T_p^*(A,B)$, then

$$\left| \left(\frac{zf'(z)}{f(z)} - p \right) \Big/ \left(B \frac{zf'(z)}{f(z)} - Ap \right) \right| < 1, \quad z \in E.$$

Since the absolute value vanishes for $z = 0$, we have

$$\left| \left(\frac{zf'(z)}{f(z)} - p \right) \Big/ \left(B \frac{zf'(z)}{f(z)} - Ap \right) \right| = h(z) \qquad \ldots(4)$$

where $h(z)$ is analytic in E and $| h(z) | < 1$ for $z \in E$. Integrating (4) with $h(z) = z\phi(z)$ we find that

$$f(z) = z^p \exp\left(p(B-A) \int_0^z \frac{\phi(t)}{1 - Bt\phi(t)}\, dt \right).$$

The converse is obtained by differentiating (3).

3. DISTORTION AND COVERING THEOREMS

Theorem 3 — If $f(z) \in T_p^*(A,B)$, then

$$r^p - \frac{(B-A)p}{1+B+(B-A)p} r^{p+1} \leqslant | f(z) | \leqslant r^p + \frac{(B-A)p}{1+B+(B-A)p} r^{p+1} \qquad \ldots(5)$$

$$pr^{p-1} - \frac{p(p+1)(B-A)}{1+B+(B-A)p} r^p \leqslant |f'(z)| \leqslant pr^{p-1} + \frac{p(p+1)(B-A)}{1+B+(B-A)p} r^p. \qquad \ldots(6)$$

The estimates are sharp.

PROOF : From Theorem 1, we have

$$[1+B+(B-A)p] \sum_{n=1}^{\infty} |a_{p+n}| \leqslant \sum_{n=1}^{\infty} [(1+B)n + (B-A)p] |a_{p+n}| \leqslant (B-A)p.$$

This implies that

$$\sum_{n=1}^{\infty} |a_{p+n}| \leqslant \frac{(B-A)p}{1+B+(B-A)p}. \qquad \ldots(7)$$

We have

$$|f(z)| \leqslant |z|^p + \sum_{n=1}^{\infty} |a_{p+n}| \, |z|^{p+n} \leqslant r^p \left(1 + r \sum_{n=1}^{\infty} |a_{p+n}|\right)$$

$$\leqslant r^p + \frac{(B-A)p}{1+B+(B-A)p} r^{p+1}.$$

and

$$|f(z)| \geqslant |z|^p - \sum_{n=1}^{\infty} |a_{p+n}| \, |z|^{p+n}$$

$$\geqslant r^p \left(1 - r \sum_{n=1}^{\infty} |a_{p+n}|\right) \geqslant r^p - \frac{(B-A)pr^{p+1}}{1+B+(B-A)p}.$$

Further

$$|f'(z)| \leqslant pr^{p-1} + \sum_{n=1}^{\infty} (p+n) |a_{p+n}| r^{p+n-1}$$

$$\leqslant pr^{p-1} + r^p \sum_{n=1}^{\infty} (p+n) |a_{p+n}|$$

$$= r^{p-1}[p + r \sum_{n=1}^{\infty} (p+n) |a_{p+n}|]. \qquad \ldots(8)$$

Similarly

$$| f'(z) | \geqslant r^{p-1}[p - r \sum_{n=1}^{\infty} (p + n) | a_{p+n} |] \qquad ...(9)$$

From Theorem 1, we have

$$\sum_{n=1}^{\infty} (1 + B)\left[n + p - \frac{p(1 + A)}{1 + B} \right] | a_{n+p} | \leqslant (B - A) p$$

or

$$\sum_{n=1}^{\infty} (1 + B) (n + p) | a_{n+p} | \leqslant (B - A) p + p(1 + A) \sum_{n=1}^{\infty} | a_{n+p} | \cdot \qquad ...(10)$$

(10) with the help of (7) implies that

$$\sum_{n=1}^{\infty} (n + p) | a_{n+p} | \leqslant \frac{(1 + p) (B - A)p}{1 + B + (B - A)p} . \qquad ...(11)$$

The estimates (6) follow from (8) and (9).

The bounds are sharp and are attained for the function

$$f(z) = z^p - \frac{(B - A)p}{1 + B + (B - A)p} z^{p+1}.$$

Corollary 3 — Let $f(z) \in T_p^*(A, B)$. Then the disc $| z | < 1$ is mapped onto a domain that contains the disc $| w | < \dfrac{1 + B}{1 + B + (B - A)p}$.

The result is sharp.

The result follows upon letting $r \to 1$ in (5).

Theorem 4 — If $f(z) = z^p - \sum_{n=1}^{\infty} | a_{n+p} | z^{n+p}$ belongs to $C_p(A, B)$, then

$$r^p - \frac{(B - A) (p + 1)}{1 + B + (B - A)} r^{p+1} \leqslant | f(z) |$$

$$\leqslant r^p + \frac{(B - A) (p + 1)}{1 + B + (B - A)p} r^{p+1} \qquad ...(12)$$

$$pr^{p-1} - \frac{(B - A) (p + 1)^2}{1 + B + (B - A)p} r^p \leqslant | f'(z) |$$

$$\leqslant pr^{p-1} + \frac{(B-A)(p+1)^2}{1+B+(B-A)p} r^p. \qquad \ldots(13)$$

The estimates are sharp for the function

$$f(z) = z^p - \frac{(B-A)(p+1)}{1+B+(B-A)p} z^{p+1}. \qquad \ldots(14)$$

The result follows from Theorem 3, as Corollary 2 follows from Theorem 1.

Corollary 4 — If $f(z) \in C_p(A, B)$, then the disc $|z| < 1$ is mapped onto a domain that contains the disc $|w| < \dfrac{1+A}{1+B+(B-A)p}$. The result is sharp for the extremal function (14).

Proof follows upon letting $r \to 1$ in (12).

4. Radius of Convexity for the Class $T_p^*(A, B)$

Theorem 5 — If $f(z) \in T_p^*(A, B)$, then $f(z)$ is p-valently convex in the disc

$$|z| < R_p = \inf_n \left[\frac{(1+B)n + (B-A)p}{(B-A)p} \left(\frac{p}{n+p}\right)^2\right]^{1/n} \quad (n = 1, 2, \ldots).$$

The result is sharp.

Proof : It is sufficient to show that $\left|\left(1 + \dfrac{zf''(z)}{f'(z)}\right) - p\right| \leqslant p$ for $|z| < R_p$.

We have

$$\left|\left(1 + \frac{zf''(z)}{f'(z)}\right) - p\right| = \left|\frac{-\sum_{n=1}^{\infty} n(n+p)|a_{n+p}| z^n}{p - \sum_{n=1}^{\infty} (n+p)|a_{n+p}| z^n}\right|$$

$$\leqslant \frac{\sum_{n=1}^{\infty} n(n+p)|a_{n+p}||z|^n}{p - \sum_{n=1}^{\infty} (n+p)|a_{n+p}||z|^n}.$$

Thus

$$\left|\left(1 + \frac{zf''(z)}{f'(z)}\right) - p\right| \leqslant p \text{ if}$$

$$\sum_{n=1}^{\infty} (n+p)^2 \; |a_{n+p}| \; |z|^n \leqslant p^2$$

or

$$\sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right)^2 \; |a_{n+p}| \; |z|^n \leqslant 1. \qquad \ldots(15)$$

By Theorem 1, $\sum_{n=1}^{\infty} \frac{(1+B)n+(B-A)p}{(B-A)p} \; |a_{n+p}| \leqslant 1$. Hence (15) will be satisfied if

$$\left(\frac{n+p}{p}\right)^2 \; |z|^n \leqslant \frac{(1+B)n+(B-A)p}{(B-A)p}$$

or if

$$|z| \leqslant \left[\frac{(1+B)n+(B-A)p}{(B-A)p}\left(\frac{p}{n+p}\right)^2\right]^{1/n} \quad (n = 1, 2, \ldots). \qquad \ldots(16)$$

The theorem follows easily from (16).

5. Closure Theorems

In this section we shall prove that the classes $T_p^*(A,B)$ and $C_p(A, B)$ are closed under convex linear combinations.

Theorem 6 — If $f(z) = z^p - \sum_{n=1}^{\infty} |a_{n+p}| z^{n+p}$ and

$$g(z) = z^p - \sum_{n=1}^{\infty} |b_{n+p}| z^{n+p}$$

are in $T_p^*(A,B)$, then $h(z) = z^p - \frac{1}{2}\sum_{n=1}^{\infty} |a_{n+p} + b_{n+p}| z^{n+p}$ is also in $T_p^*(A,B)$.

Proof : Since $f(z)$ and $g(z)$ are in $T_p^*(A,B)$, we have

$$\sum_{n=1}^{\infty} [(1+B)n + (B-A)p] \, |a_{n+p}| \leqslant (B-A)p \qquad \ldots(17)$$

and

$$\sum_{n=1}^{\infty} [(1+B)n + (B-A)p] \, |b_{n+p}| \leqslant (B-A)p. \qquad \ldots(18)$$

From (17) and (18) we get

$$\frac{1}{2} \sum_{n=1}^{\infty} [(1 + B)n + (B - A)p] \mid a_{n+p} + b_{n+p} \mid \leqslant (B - A)p$$

which implies that $h(z) \in T_p^*(A, B)$.

The following theorem can be proved similarly.

Theorem 7 — If $f(z) = z^p - \sum_{n=1}^{\infty} \mid a_{n+p} \mid z^{n+p}$ and $g(z) = z^p - \sum_{n=1}^{\infty} \mid b_{n+p} \mid z^{n+p}$ are in $C_p(A, B)$, then $h(z) = z^p - \frac{1}{2} \sum_{n=1}^{\infty} \mid a_{n+p} + b_{n+p} \mid z^{n+p}$ is also in $C_p(A, B)$.

Theorem 8 — Let $f_p(z) = z^p, f_{n+p}(z) = z^p - \frac{(B - A)p}{(1 + B)n + (B - A)p} z^{n+p}$

$(n = 1, 2, 3 \ldots)$.

Then $f \in T_p^*(A, B)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=0}^{\infty} \lambda_{n+p} f_{n+p}(z) \text{ where } \lambda_{n+p} \geqslant 0 \text{ and } \sum_{n=0}^{\infty} \lambda_{n+p} = 1.$$

PROOF : Suppose $f(z) = \sum_{n=0}^{\infty} \lambda_{n+p} f_{n+p}(z)$

$$= z^p - \sum_{n=1}^{\infty} \frac{(B - A)p}{(1 + B)n + (B - A)p} \lambda_{n+p} z^{n+p}.$$

Then
$$\sum_{n=1}^{\infty} \left[\lambda_{n+p} \frac{(1 + B)n + (B - A)p}{(B - A)p} \left(\frac{(B - A)p}{(1 + B)n + (B - A)p} \right) \right]$$

$$= \sum_{n=1}^{\infty} \lambda_{n+p} = 1 - \lambda_p \leqslant 1.$$

So by Theorem 1, $f(z) \in T_p^*(A, B)$.

Conversely, let $f(z) \in T_p^*(A, B)$. Then

$$\mid a_{n+p} \mid \leqslant \frac{(B - A)p}{(1 + B)n + (B - A)p}.$$

Setting $$\lambda_{n+p} = \frac{(1+B)n + (B-A)p}{(B-A)p} \mid a_{n+p} \mid \qquad (n = 1, 2, \ldots),$$

and

$$\lambda_p = 1 - \sum_{n=1}^{\infty} \lambda_{n+p},$$

we have

$$f(z) = \sum_{n=0}^{\infty} \lambda_{n+p} f_{n+p}(z).$$

This completes the proof of the theorem.

Remarks: (i) Putting $p = 1$ and taking $A = (2\alpha - 1)\,\beta$, $B = \beta$, where $0 \leqslant \alpha < 1$ and $0 < \beta \leqslant 1$, in the above theorems we get the results obtained by Gupta and Jain (1976). (ii) On further taking $\beta = 1$, we obtain the results due to Silverman (1975).

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