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A NOTE ON THE SPACE OF INTEGRAL FUNCTIONS

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In this note a sufficient condition is obtained for $L\{\alpha^{(n)}, n \geqslant 0\}$ to be the whole of Γ.

§1. The space Γ of integral functions $\alpha = \alpha(z) = \sum_0^\infty a_n z^n$ with the metric $|\alpha - \beta|$, $\alpha, \beta \in \Gamma$ where $|\alpha| = \text{Sup}\{|a_0|, |a_n|^{1/n}, n \geqslant 1\}$ is known to be a complete linear metric space (Iyer 1948). Its topological dual Γ^* can be identified with the space of all power series $f = f(z) = \sum_0^\infty c_n z^n$ with positive radius of convergence, that is, $|c_n|^{1/n}$ bounded and $f(\alpha) = \sum_0^\infty c_n a_n$ (Iyer 1948).

Let $E \subset \Gamma$. We shall denote by $L(E)$ the closed linear subspace generated by E, that is, the closure of the linear subspace consisting of finite linear combinations of elements taken from E. Since the space Γ has the Hahn-Banach property of normed linear spaces (Iyer 1950), it follows that $\alpha \in L(E)$ if and only if every $f \in \Gamma^*$ vanishing on E also vanishes at α. Let $\alpha \in \Gamma$ and Z a set of complex numbers. Let $L\{\alpha, Z\}$ denote the closed linear subspace generated by $E = \{\alpha(z + \lambda), \lambda \in Z\}$. We have considered elsewhere (Iyer 1952), conditions on α and Z in order that

$$L\{\alpha, Z\} = L\{\alpha^{(n)}, n \geqslant 0\}$$

where $\alpha^{(0)} = \alpha$ and $\alpha^{(n)} =$ the nth derivative of α.

In this note a sufficient condition is obtained for $L\{\alpha^{(n)}, n \geqslant 0\}$ to be the whole of Γ. The result is contained in the following theorem.

§2. *Theorem* : Let $\alpha \in \Gamma$ be a transcendental integral function of order one and minimal type, that is, $\log M(r, \alpha)/r$ tends to 0 as r tends to infinity where $M(r, \alpha) = \max_{|z|=r} |\alpha(z)|$. Then $L\{\alpha^{(n)}, n \geqslant 0\} = \Gamma$.

The proof of the result is based on the following lemma.

Lemma — Let $\alpha(z)$ be regular near $z = 0$ and $\beta(z)$ a transcendental integral function. Then $\alpha(z) . \beta(1/z)$ can be regular near $z = 0$ if and only if $\alpha(z)$ is identically equal to zero.

PROOF : Let $\alpha(z) \not\equiv 0$. Then there is a first coefficient $a_k \neq 0$. We can write $\alpha(z) \cdot \beta(1/z) = \alpha_1(z) \{z^k \cdot \beta(1/z)\}$ where $\alpha_1(0) = a_k \neq 0$. Since β is transcendental, $z^k \cdot \beta(1/z)$ has an essential singularity at $z = 0$. So by Weierstrass' theorem, given any complex number w, there is a sequence $\{z_p\}$ converging to zero so that $\alpha(z) \cdot \beta(1/z)$ tends to $a_k \cdot w$ as $z = z_p$ tends to 0. This limit varies with w since $a_k \neq 0$ and this is a contradiction since $\alpha(z) \cdot \beta(1/z)$ being regular at $z = 0$ tends to a definite limit as z tends to 0. This proves the lemma.

Proof of the Theorem — By the result quoted in §1.1, it is enough to prove that if an $f = \sum_0^\infty c_n z^n \in \Gamma^*$ vanishes on $\{\alpha^{(n)}, n \geqslant 0\}$, it is identically zero. Taking $\alpha = \sum_0^\infty a_n z^n/n!$, $f(\alpha^{(k)}) = 0$ for $k = 0, 1, 2, \ldots$ implies

$$(*) \quad \sum_{n=0}^{\infty} (c_n/n!) \cdot a_{n+k} = 0, \; k = 0, 1, 2, \ldots$$

Also since α is of order one and minimal type, by a well known result, $n \cdot | a_n/n! |^{1/n}$ tends to 0 as n tends to infinity and since $n/(n!)^{1/n}$ tends to e as n tends to infinity, it follows that $| a_n |^{1/n}$ tends to zero as n tends to infinity. Hence $\beta(z) = \sum_0^\infty a_n z^n$ is a transcendental integral function. Also since $| c_n |^{1/n}$ is bounded, $\gamma(z) = \sum_0^\infty c_n z^n/n!$ is an integral function. The relations (*) imply that the product $\gamma(z) \cdot \beta(1/z)$ is regular near $z = 0$. This, by the lemma, implies $\gamma(z) \equiv 0$ or $c_n = 0, n = 0, 1, 2, \ldots$. This completes the proof.

Remark: The result of the theorem is sharp in the sense that there are functions of order one and positive type for which the theorem fails. For instance, we can take $\alpha(z) = \exp(az)$, $a \neq 0$. But the condition is not necessary. For instance, $\alpha(z) = \exp(z^2)$ is of order two and type one. Here $L\{\alpha^{(n)}, n \geqslant 0\} = \Gamma$ (Iyer 1952).

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