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ON THE REPRESENTATION OF FUNCTIONS AS GENERALIZED LAPLACE INTEGRALS

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In this paper we obtain the conditions, necessary and sufficient, in order that a function be the generalized Laplace transform [discussed by Joshi (1963) in detail], where the determining function belongs to the Orlicz class.

1. INTRODUCTION

Joshi (1963) has given a generalization of Laplace transform

$$f(x) = \int_0^\infty e^{-xt}\phi(t)\, dt \qquad \ldots(1.1)$$

in the form,

$$F(x) = \frac{\Gamma(\beta + \eta + 1)}{\Gamma(\alpha + \beta + \eta + 1)} \int_0^\infty [{}_1F_1(\beta + \eta + 1,\ \alpha + \beta + \eta + 1; -xt) \times (xt)^\beta\, \phi(t)\, dt] \qquad \ldots(1.2)$$

for $x > 0$, Re $\beta \geqq 0$, Re $\eta > 0$, and has discussed this transform in detail. Here we shall try to find conditions, necessary and sufficient, in order that $F(x)$ $(0 < x < \infty)$ should possess the representation (1.2) where $\{t^\beta \phi(t)\}$ belongs to a given class of functions measurable in $0 \leqq t < \infty$,

Let $M(u)$ be an even, complex, continuous function, satisfying

(i) $M(u)/u \to 0 \quad (u \to 0)$,

(ii) $M(u)/u \to \infty \quad (u \to \infty)$.

Denote by $L_M\,[0, \infty]$, the class of all functions $g(x)$, measurable in $0 \leqq x < \infty$, such that $\int_0^\infty M[g(x)]\, dx < \infty$.

$L_M\,[0, \infty]$ is the Orlicz class related to $M(u)$. (Kransnosel'skii and Rutickii 1961), If we take $M(u) = |\,u\,|^p$, $1 < p < \infty$, then $L_M\,[0, \infty]$ is the space $L^p\,[0,\infty]$.

2. Representation Theorem

In this section we give representation theorem for (1. 2) under conditions to be specified presently.

Theorem — Necessary and sufficient conditions, in order that $F(x)$ should possess the representation (1.2) where $\{t^{\beta}\phi(t)\} \in L_M\,[0, \infty]$, are:

(i) $F(x)$ has derivatives of all orders in $0 < x < \infty$,

(ii)
$$\sup_{0<x<\infty} \sum_{k=0}^{\infty} \frac{(b-1)}{(a-1)} \frac{1}{x} M\left[\frac{x^{k+1}}{\Gamma(k+1)} \frac{(a-1)}{(b-1)} D^{k} \left\{\frac{\Gamma(b)}{\Gamma(a)} x^{-\beta} F(x)\right\}\right] \equiv H \qquad \ldots(2.1)$$

$$\left(\text{where} \quad \beta + \eta + 1 = a, \quad \alpha + \beta + \eta + 1 = b \quad \text{and} \quad D \equiv \frac{d}{d\alpha}\right)$$

provided that,

$$1 < \operatorname{Re}(\beta + \eta + 1), \ \operatorname{Re}\alpha \leqq 0, \operatorname{Re}(\alpha + \beta + \eta + 1) > 0.$$

Proof : *Necessity* : First suppose that $F(x)$ possesses the representation (1.2) where $\{t^{\beta}\phi(t)\} \in L_M\,[0, \infty]$ and we shall prove that the integral on the right-hand side of (1.2) converges absolutely for $x > 0$, $\operatorname{Re}\beta \geqq 0$, $\operatorname{Re}\eta > 0$. Let $N(v)$ be the complement function to $M(u)$ (see Kransnosel'skii and Rutickii 1961, pp. 11, 12), then by Young's inequality and (Erdelyi 1953, p. 285)

$$| F(x) | \leqq \frac{\Gamma(a)}{\Gamma(b)} \int_{0}^{\infty} {}_1F_1(a, b; -xt)\, x^{\beta} \ | t^{\beta}\phi(t) | \, dt$$

$$\leqq \int_{0}^{\infty} M\,\{t^{\beta}\phi(t)\}\, dt + \int_{0}^{\infty} N\left\{\frac{\Gamma(a)}{\Gamma(b)} x^{\beta}\, {}_1F_1(a, b; -xt)\right\} dt$$

$$\leqq \int_{0}^{\infty} M\,\{t^{\beta}\phi(t)\}\, dt + N(1)\, x^{\beta-1} \frac{\Gamma(a-1)}{\Gamma(b-1)}$$

[since for $0 < v < 1$, $N(v) \leqq v \,.\, N(1)$ and $0 < 1 < \operatorname{Re}(\beta + \eta + 1)$]

$$< \infty.$$

Hence, $F(x)$ has derivatives of all orders in $0 < x < \infty$ and by (Erdelyi 1953, p. 254)

$$\psi^{(k)}(x) = \frac{(a)_k}{(b)_k} (-1)^k \int_{0}^{\infty} \left[{}_1F_1(a+k, b+k; -xt),\, t^{\beta+k}\phi(t)\, dt\right] \qquad \ldots(2.2)$$

for $x > 0$, $\operatorname{Re}\beta \geqq 0$, $\operatorname{Re}\eta > 0$.

where $\psi(x) = \frac{\Gamma(b)}{\Gamma(a)} x^{-\beta} F(x).$

Now, by (2.2) and Jensen's inequality and Erdelyi (1953, p. 285)

$$M\left\{\frac{x^{k+1}}{\Gamma(k+1)} \frac{(a-1)}{(b-1)} \psi^{(k)}(x)\right\} = M\left\{\frac{\frac{(a)_k}{(b)_k}\int_0^\infty [{}_1F_1(a+k, b+k; -xt), t^{\beta+k}\phi(t)\,dt]}{\frac{(a)_k}{(b)_k}\int_0^\infty [{}_1F_1(a+k, b+k; -xt), t^k dt]}\right\}$$

where $0 < k+1 < \text{Re}(\beta + \eta + k + 1)$,

$$\leqq \frac{\frac{(a)_k}{(b)_k}\int_0^\infty [{}_1F_1(a+k, b+k; -xt)\, t^k M\{t^\beta \phi(t)\}\,dt]}{\frac{(a)_k}{(b)_k}\int_0^\infty [{}_1F_1(a+k, b+k; -xt)\, t^k dt]}$$

$$= \frac{x(a-1)}{(b-1)} \int_0^\infty \left[\frac{(a)_k}{(b)_k} \frac{1}{\Gamma(k+1)}\, {}_1F_1(a+k, b+k; -xt)\,(xt)^k M\{t^\beta \phi(t)\}\,dt\right].$$

Hence, by Levi's theorem and (Erdelyi 1953, p. 283)

$$\sum_{k=0}^{\infty} \frac{1}{x} \frac{(b-1)}{(a-1)} M\left\{\frac{x^{k+1}}{\Gamma(k+1)} \frac{(a-1)}{(b-1)} \psi^{(k)}(x)\right\}$$

$$\leqq \int_0^\infty M\{t^\beta \phi(t)\}\,dt \equiv H. \qquad \ldots(2.3)$$

This proves the necessity.

Sufficiency — In order to prove sufficiency, let the conditions (i) and (ii) hold. By Taylor's expansion with remainder, we have

$$\psi(x) = \sum_{n=0}^{k} \psi^{(n)}(a') \frac{(x-a')^n}{n!} + \psi^{(k+1)}(\xi) \frac{(x-a')^{k+1}}{(k+1)!}$$

$$(0 < x < \xi < a')$$

Also, by eqn. (2.1),

$$\frac{(b-1)}{(a-1)}\frac{1}{x}M\left\{\frac{x^{k+2}}{\Gamma(k+2)}\frac{(a-1)}{(b-1)}\psi^{(k+1)}(x)\right\} \leqq H \qquad \ldots(2.4)$$

for $0 < x < \infty$.

Now, by (2.4) and Young's inequality

(Kransnosel'skii and Rutickii 1961, p. 12), we have,

$$\left|\psi^{(k+1)}(\xi)\frac{(x-a')^{k+1}}{\Gamma(k+2)}\right| \leqq \left(\frac{|x-a'|}{\xi}\right)^{k+1}\left[\frac{(b-1)}{(a-1)}\frac{1}{\xi}M\left\{\frac{(a-1)}{(b-1)}\right.\right.$$

$$\left.\left.\times\frac{\xi^{k+2}}{\Gamma(k+2)}\psi^{(k+1)}(\xi)+N(1)\,\xi^{-1}\frac{(b-1)}{(a-1)}\right]$$

$$\leqq \left(\frac{|x-a'|}{\xi}\right)^{k+1}[H+N(1)\,\xi^{-1}] \qquad \ldots(2.5)$$

if $\operatorname{Re}\alpha \leqq 0$.

Now, if $a'/2 < x \leqq a'$, then

$$\left|\psi^{(k+1)}(\xi)\frac{(x-a')^{k+1}}{\Gamma(k+2)}\right| \leqq \left(\frac{2a'-2x}{a'}\right)^{k+1}[H+2N(1)\,a'^{-1}]$$

$$= O(1) \qquad (k\to\infty).$$

Hence the series $\displaystyle\sum_{k=0}^{\infty}\psi^{(k)}(a')\frac{(x-a')^k}{k!}$ converges to $\psi(x)$ for $a'/2 < x \leqq a'$. Since a' is arbitrary, $\psi(x)$ is analytic in $0 < x < \infty$.

Now, it follows from (2.1) and the analyticity of $\psi(x)$, that for every $u > 0$, $0 < x < \infty$,

$$\psi(u) = \lim_{x\to\infty}\sum_{k=0}^{\infty}{}_1F_1\left(a, b; \frac{-ku}{x}\right)\frac{(-x)^k}{\Gamma(k+1)}\psi^{(k)}(x) \qquad \ldots(2.6)$$

$$\left[\text{For, } \left|\psi(u)-\sum_{k=0}^{\infty}{}_1F_1\left(a, b; \frac{-ku}{x}\right)\frac{(-x)^k}{\Gamma(k+1)}\psi^{(k)}(x)\right|\right.$$

$$\left.\leqq \sum_{k=0}^{\infty}\left[\frac{x^k}{\Gamma(k+1)}|\psi^{(k)}(x)|\left|\left(1-\frac{u}{x}\right)^k-{}_1F_1\left(a, b; \frac{-ku}{x}\right)\right|\right]\right].$$

By (2.5) we see that $\dfrac{x^k}{\Gamma(k+1)}|\psi^{(k)}(x)|$ is bounded (i.e. $\leqq [H+N(1)\,x^{-1}]$) for $0 < x < \infty$, $\operatorname{Re}\alpha \leqq 0$ and the remaining terms in the right-hand side tend to 0 as x tends to ∞. Hence (2.6) is proved. Let us set the functions $\alpha_x(t)$ as:

$$\alpha_x(0) = 0,\ \alpha_x(t) = \sum_{k=0}^{[xt]} \frac{(-x)^k}{\Gamma(k+1)} \psi^{(k)}(x) \qquad 0 < t < \infty. \qquad \ldots(2.7)$$

Then by Young's inequality for every fixed R, $0 < R < \infty$,

$$\int_0^R |\, d\alpha_x(t) \,| \leqq \sum_{k=0}^{[xR]} \frac{x^k}{\Gamma(k+1)} |\, \psi^{(k)}(x) \,|$$

$$\leqq \sum_{k=0}^{[xR]} \left[\frac{(b-1)}{(a-1)} \frac{1}{x} M \left\{ \frac{x^{k+1}}{\Gamma(k+1)} \psi^{(k)}(x) \frac{(a-1)}{(b-1)} \right\} + N(1)\, x^{-1} \frac{(b-1)}{(a-1)} \right]$$

$$\leqq H + N(1)\,(R+1) \qquad (\text{since } \operatorname{Re} \alpha \leqq 0). \qquad \ldots(2.8)$$

Hence, the functions $\alpha_x(t)$ $(x > 0)$ are of uniformly bounded variations in $[0, R]$ for every R, $0 < R < \infty$.

Also, by (2.8)

$$|\, \alpha_x(t) \,| \leqq \int_0^t |\, d\alpha_x(u) \,| \leqq H + N(1)\,(t+1). \qquad \ldots(2.9)$$

Let $u > 0$, then by (2.6) and (2.7) it is clear that $\int_0^\infty {}_1F_1(a, b; -ut)\, d\alpha_x(t)$ exists for every x, $0 < x < \infty$ and $\operatorname{Re} a > 1$, $\operatorname{Re} b > 0$, $\operatorname{Re} \alpha \leqq 0$, $u > 0$ and

$$\psi(u) = \lim_{x\to\infty} \int_0^\infty {}_1F_1(a, b; -ut)\, d\alpha_x(t)$$

$$= \lim_{x\to\infty} \frac{a}{b}\, u \int_0^\infty {}_1F_1(a+1, b+1; -ut)\, \alpha_x(t)\, dt$$

for $u > 0$, $\operatorname{Re} a > 1$, $\operatorname{Re} b > 0$, $\operatorname{Re} \alpha \leqq 0$.

Now, by Helly's theorem (Widder 1946, p. 28) (since $|\, \alpha_x(t) \,| < A$) there exists a sequence $\{x_n\}$, $x_n \to \infty$ and a function $\alpha(t)$, such that

$$\lim_{x_n\to\infty} \alpha_{x_n}(t) = \alpha(t); \qquad 0 < t < \infty.$$

Therefore by Lebesgue's theorem on dominant convergence (Titchmarsh 1939, p. 337),

$$\psi(u) = \frac{a}{b} u \int_0^\infty {}_1F_1(a+1, b+1; -ut)\, \alpha(t)\, dt$$

$$= \int_0^\infty {}_1F_1(a, b; -ut)\, d\alpha(t)$$

for $u > 0$, Re $a > 1$, Re $b > 0$, Re $\alpha \leqq 0$

or,
$$F(u) = \frac{\Gamma(a)}{\Gamma(b)} \int_0^\infty {}_1F_1(a, b; -ut)\, u^{\beta} d\alpha(t) \qquad \text{...(2.10)}$$

for $u > 0$, Re $\beta \geqq 0$, Re $\eta > 0$, $-1 \leqq$ Re $\alpha \leqq 0$.

Let $0 = t_0 < t_1 < \ldots < t_k < \infty$ and $[xt_i]$ be denoted by x_i, then

$$M\left\{\frac{(a-1)}{(b-1)} \frac{\alpha_x(t_{i+1}) - \alpha_x(t_i)}{x^{-1}(x_{i+1} - x_i)}\right\}$$

$$= M\left\{\frac{\displaystyle\sum_{k=x_i+1}^{x_{i+1}} \frac{(-x)^{k+1}}{\Gamma(k+1)} \frac{(a-1)}{(b-1)} \psi^{(k)}(x)}{x_{i+1} - x_i}\right\}$$

$$\leqq \sum_{k=x_i+1}^{x_{i+1}} \left[M\left\{\frac{(a-1)}{(b-1)} \frac{x^{k+1}}{\Gamma(k+1)} \psi^{(k)}(x)\right\} \Big/ (x_{i+1} - x_i)\right]$$

[By Jensen's inequality, see Zygmund (1959), p. 24]

Hence by (2.1)

$$\sum_{i=0}^{k-1} \frac{(b-1)}{(a-1)} \frac{x_{i+1} - x_i}{x} M\left\{\frac{(a-1)}{(b-1)} \frac{\alpha_x(t_{i+1}) - \alpha_x(t_i)}{x^{-1}(x_{i+1} - x_i)}\right\}$$

$$\leqq \sum_{k=0}^{\infty} \frac{(b-1)}{(a-1)} \frac{1}{x} M\left\{\frac{(a-1)}{(b-1)} \frac{x^{k+1}}{\Gamma(k+1)} \psi^{(k)}(x)\right\}$$

$$\leqq H. \qquad \text{...(2.11)}$$

Since,

$$\lim_{n\to\infty} \{\alpha_{x_n}(t_{i+1}) - \alpha_{x_n}(t_i)\} = \alpha(t_{i+1}) - \alpha(t_i)$$

and

$$\lim_{x\to\infty} \left(\frac{x_{i+1} - x_i}{x}\right) = t_{i+1} - t_i,$$

we have

$$\sum_{i=0}^{k-1} \frac{(b-1)}{(a-1)} (t_{i+1} - t_i) \, M \left\{ \frac{\alpha(t_{i+1}) - \alpha(t_i)}{(t_{i+1} - t_i)\,(b-1)/(a-1)} \right\} \leqq H \quad ...(2.12)$$

(2.12) holds for every

$$0 = t_0 < t_1 < \ldots < t_k < \infty,$$

therefore, an easy modification of Medvedev's theorem (see Medvedev 1953) gives

$$\alpha(t) = c + \int_0^t v^{\beta} \phi(v) \, dv \quad ...(2.13)$$

where

$$\{v^{\beta}\phi(v)\} \in L_M \, [0, \infty].$$

By (2.13) and (2.10), we have

$$F(u) = \frac{\Gamma(a)}{\Gamma(b)} \int_0^{\infty} {}_1F_1(a, b; -ut) \, (ut)^{\beta} \, \phi(t) \, dt$$

for $\operatorname{Re} \beta \geqq 0$, $\operatorname{Re} \eta > 0$, $-1 \leqq \operatorname{Re} \alpha \leqq 0$, $u > 0$.

Hence the theorem.

Corollary — Necessary and sufficient conditions, in order that $F(x)$ should possess the representation (1.2) where $\{t^{\beta}\phi(t)\} \in L^{p} \, [0, \infty]$, are:

(i) $F(x)$ has derivatives of all orders in $0 < x < \infty$,

(ii) $$\sup_{0<x<\infty} \left\{ \frac{(a-1)}{(b-1)} x \right\}^{p-1} \left| \frac{x^k}{\Gamma(k+1)} D^k \left[\frac{\Gamma(b)}{\Gamma(a)} x^{-\beta} F(x) \right] \right|^{p} \equiv H < \infty,$$

provided that

$$\operatorname{Re}(\beta + \eta + 1) > 1, \ \operatorname{Re} \alpha \leqq 0, \ \operatorname{Re}(\alpha + \beta + \eta + 1) > 0.$$

3. Another Theorem

Theorem — If $F(x)$ possesses the representation (1.2) where $\{t^{\beta}\phi(t)\} \in L_M \, [0, \infty]$ then

$$\lim_{x \to \infty} \sum_{k=0}^{\infty} \frac{(b-1)}{(a-1)} \frac{1}{x} \, M \left\{ \frac{x^{k+1}}{\Gamma(k+1)} \frac{(a-1)}{(b-1)} D^k \left[\frac{\Gamma(b)}{\Gamma(a)} x^{-\beta} F(x) \right] \right\}$$

$$= \int_0^{\infty} M\{t^{\beta}\phi(t)\} \, dt, \quad ...(3.1)$$

provided that

$$\operatorname{Re}(\beta + \eta + 1) > 1, \operatorname{Re} \alpha \leqq 0, \operatorname{Re}(\alpha + \beta + \eta + 1) > 0.$$

The constants a and b are as mentioned before.

PROOF : If $F(x)$ possess the representation (1.2),

where $\{t^{\beta}\phi(t)\} \in L_M[0, \infty]$, then by (2.3),

$$\lim_{x \to \infty} \sum_{k=0}^{\infty} \frac{(b-1)}{(a-1)} \frac{1}{x} M \left\{ \frac{x^{k+1}}{\Gamma(k+1)} \frac{(a-1)}{(b-1)} \psi^{(k)}(x) \right\}$$

$$\leqq \int_0^{\infty} M\{t^{\beta}\phi(t)\}\, dt, \text{ if } 0 < 1 < \operatorname{Re} a,\ k+1 < \operatorname{Re}(a+k). \quad \ldots(3.2)$$

[where $\psi(x) = (\Gamma b\ \Gamma a)\, x^{-\beta} F(x)$]

Also, in the previous theorem we have proved that if $\alpha_x(t)$ is defined by (2.7), then by Helly's the orem (Widder 1946, p. 26) there exists a subsequence $\{y_k\}$, $(k \geqq 0)$ for every sequence $\{x_n\}$, $x_n \to \infty$, such that

$$\lim_{k \to \infty} \alpha_{y_k}(t) = \alpha(t), \qquad 0 < t < \infty$$

and

$$F(u) = \frac{\Gamma(a)}{\Gamma(b)} \int_0^{\infty} {}_1F_1(a, b; -ut)\, u^{\beta} d\alpha(t)$$

for $u > 0$, $\operatorname{Re}\beta \geqq 0$, $\operatorname{Re}\eta > 0$, $-1 \leqq \operatorname{Re}\alpha \leqq 0$ [see eqn. (2.10)].

By the uniqueness of determining function of a Laplace integral and by (2.13), we have

$$\lim_{x \to \infty} \alpha_x(t) = c + \int_0^{t} v^{\beta}\phi(v)\, dv, \quad \ldots(3.3)$$

Hence by eqns. (2.11), (3.3) and by the proof of Medvedev's (1953) theorem we obtain,

$$\int_0^{\infty} M\{t^{\beta}\phi(t)\}\, dt \leqq \liminf_{x \to \infty} \sum_{k=0}^{\infty} \left[\frac{(b-1)}{(a-1)} \frac{1}{x} \right.$$

$$\left. \times M \left\{ \frac{(a-1)}{(b-1)} \frac{x^{k+1}}{\Gamma(k+1)} \psi^{(k)}(x) \right] \right\}. \quad \ldots(3.4)$$

By eqns. (3.2) and (3.4) we have

$$\lim_{x\to\infty}\sum_{k=0}^{\infty}\frac{(b-1)}{(a-1)}\frac{1}{x}M\left\{\frac{x^{k+1}}{\Gamma(k+1)}\frac{(a-1)}{(b-1)}D^{k}\left[\frac{\Gamma(a)}{\Gamma(b)}x^{-\beta}F(x)\right]\right\}$$

$$=\int_0^\infty M\{t^{\beta}\phi(t)\}\,dt$$

(where $a=\beta+\eta+1$, $b=\alpha+\beta+\eta+1$)

thus the proof is complete.

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