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BORNOLOGICAL PROPERTIES OF THE SPACE OF INTEGRAL FUNCTIONS

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In this paper we study the bornological aspect of the space Γ of integral functions over the complex field C. By $\bar{\Gamma}$ we denote the space of all power series with positive radius of convergence at $z = 0$. We introduce bornologies on Γ and $\bar{\Gamma}$ and prove that $\bar{\Gamma}$ is a convex bornological vector space which is the completion of the convex bornological vector space Γ.

1. INTRODUCTION

Iyer (1948, 1950) has studied the space Γ of integral functions over the complex field C. He introduced a real-valued map on Γ and proved that this map defines a metric on Γ. In this paper we study the bornological aspect of some of his results on the space Γ. For definitions and notations in bornology the reader is directed to Hogbe-Nlend (1971).

2. THE SPACE Γ

Γ denotes the vector space of all integral functions over the complex field C. It is known that $\alpha \in \Gamma$ if and only if $\alpha = a_0 + a_1 z + \dots + a_n z^n + \dots$ where $a_i \in C$, $i \geqslant 0$ and $\lim_{n \to \infty} | a_n |^{1/n} = 0$. For each $\alpha \in \Gamma$ there is an associated real number defined by

$$\| \alpha \| = \text{Sup} \{ | a_0 | , | a_n |^{1/n}, n \geqslant 1\}$$

satisfying

1. $\| \alpha \| \geqslant 0$ and $\| \alpha \| = 0$ if and only if $\alpha = \theta$

   where $\theta = \Sigma a_n z^n$, $a_i = 0$ for all $i \geqslant 0$;

2. $\| \alpha + \beta \| \leqslant \| \alpha \| + \| \beta \|$;

3. $\| \lambda\alpha \| \leqslant A(\lambda) \| \alpha \|$ where $A(\lambda) = \max (1, | \lambda |), \lambda \in C$.

We define a bornology on Γ with the help of this $\| \alpha \|$. We denote by B_r the set $\{\alpha \in \Gamma \mid \| \alpha \| \leqslant r\}$. Then the family $\mathcal{B}' = \{B_r \mid r = 1, 2, \dots\}$ forms a base for a bornology $\mathcal{B}$ on Γ.

Theorem 2.1 — $(\Gamma, \mathscr{B})$ is a separated convex bornological vector space with a countable base.

The proof of the Theorem 2.1 is straightforward.

In the sequel we shall mean by a bounded set a set bounded in the sense of bornology, unless stated to the contrary.

Definition 1 — A set P is said to be bornivorous if for every bounded set B there exists a $\lambda \in C$, $\lambda \neq 0$ such that $\mu B \subset P$ for all $\mu \in C$ for which $|\mu| \leqslant |\lambda|$.

Theorem 2.2 — $\mathscr{B}$ contains no bornivorous set.

PROOF : Suppose $\mathscr{B}$ contains a bornivorous set A. Then there exists a $B_i \in \mathscr{B}$ such that $A \subset B_i$ and consequently B_i is also bornivorous. We now assert that if $i_1 > i$, then $\lambda B_{i_1} \not\subset B_i$ for any $\lambda \in C$ which leads to a contradiction. If $i_1 > i$, it is easy to see that $\lambda B_{i_1} \not\subset B_i$ for any $\lambda \in C$ such that $|\lambda| \geqslant 1$. Now we prove that $\lambda B_{i_1} \not\subset B_i$ for any $\lambda \in C$ such that $|\lambda| < 1$ also.

Let thus $|\lambda| < 1$. Since $i_1/i > 1$, we can choose n such that $1 < 1/|\lambda| < (i_1/i)^n$. Now let $a_n \in C$ be such that $i^n/|\lambda| < |a_n| \leqslant i_1^n$ and let $\alpha = a_n z^n$. Then $\|\alpha\| = |a_n|^{1/n} \leqslant i_1$ and hence $\alpha \in B_{i_1}$. Now $\|\lambda\alpha\| = \|\lambda a_n z^n\| = |\lambda a_n|^{1/n} > i$ and hence $\lambda\alpha \notin B_i$, Thus $\lambda B_{i_1} \not\subset B_i$ for any $\lambda \in C$.

Definition 2 — Let E be a bornological vector space. A sequence $\{x_n\}$ in E is said to M-converge to a point $x \in E$ if there exists a sequence $\{\lambda_n\}$ of decreasing positive real numbers tending to zero such that the sequence $\left\{\frac{x_n - x}{\lambda_n}\right\}$ is bounded.

Theorem 2.3 — The M-convergence in a bornological vector space E is topologisable if and only if E has a bounded bornivorous set.

For proof of Theorem 2.3 see Hogbe-Nlend (1971, Proposition 1, p. 12).

Corollary 2.1 — The M-convergence of Γ is not topologisable.

PROOF : Suppose the M-convergence of Γ is topologisable. Then Γ has a bounded bornivorous set by Theorem 2.3 and this contradicts Theorem 2.2.

3. THE BORNOLOGICAL DUAL OF Γ

Iyer (1948) has proved that a linear functional f on Γ is of the form $f = \sum_{0}^{\infty} c_n z^n$ where $c_n \in C$ and $\{|c_n|^{1/n}, n \geqslant 1\}$ is bounded. Further, if $\alpha \in \Gamma$, $\alpha = \Sigma a_n z^n$ then $f(\alpha) = \Sigma c_n a_n$.

Lemma 3.1 — A linear functional $f: \Gamma \to C$ is bounded if and only if f maps every M-convergent sequence to a bounded sequence in C.

PROOF : See Hogbe-Nlend (1971, Proposition 1, p. 10)

Theorem 3.1 — A linear functional $f = \Sigma c_n z^n$ on Γ is bounded if and only if $\lim_{n\to\infty} |c_n|^{1/n} = 0$ i.e. $f \in \Gamma$.

PROOF : *If part* — Suppose $|c_n|^{1/n} \to 0$. Let $\{\alpha_q\}$ be a sequence in Γ such that $\alpha_q \xrightarrow{M} 0$. Then there exists a constant k and a decreasing sequence $\{\lambda_q\}$ of scalars converging to zero such that $\| \alpha_q/\lambda_q \| \leqslant k$.

i.e. $$|a_{q,0}| \leqslant |\lambda_q| k \text{ and } |a_{q,n}| \leqslant |\lambda_q| k^n, n \geqslant 1.$$

Since $|c_n|^{1/n} \to 0$, there exists n_0 such that $|c_n|^{1/n} \leqslant \frac{1}{2k}$ for all $n \geqslant n_0$. Hence $|c_n| \leqslant 1/(2k)^n$, $(n \geqslant n_0)$.

Now,
$$\begin{aligned} |f(\alpha_q)| &= |\Sigma c_n a_{q,n}| \leqslant \Sigma |c_n| \; |a_{q,n}| \\ &\leqslant \sum_0^\infty |c_n| \; |\lambda_q| k^n \\ &\leqslant \sum_0^{n_0-1} |c_n| \; |\lambda_q| k^n + \sum_{n_0}^\infty |\lambda_q| /2^n \\ &< \infty \text{ (independent of } q). \end{aligned}$$

Thus $\{f(\alpha_q)\}$ is bounded. Hence f is bounded on every sequence which M-converges to zero and consequently by Lemma 3.1 f is bounded.

Only if part — Let $f = \Sigma c_n z^n$ be such that $\overline{\lim_{n\to\infty}} |c_n|^{1/n} = \rho \neq 0$. Then given $\eta > 0$ such that $\eta < \rho$, there exists a divergent increasing sequence $\{n_q\}$ of integers such that $|c_n|^{1/n} > \eta$ for all $n = n_q$. Choose $\pi \in R$ such that $\pi > 1$ and $\pi \eta > 1$. Consider $\{\alpha_n\}$ where $\alpha_n = \pi^n z^n \in \Gamma$ and define $\lambda_n \in C$ as

$$\lambda_n = 1/\pi^n.$$

Then $\lambda_n \to 0$ and

$$\| \alpha_n/\lambda_n \| = \| \pi^{2n} z^n \| = |\pi|^2 < \infty.$$

Consequently $\alpha_n \xrightarrow{M} 0$. But $f(\alpha_n) = c_n \pi^n$ and

$$|f(\alpha_{n_q})| = |c_{n_q}| \; |\pi|^{nq} > \eta^{nq} |\pi|^{nq}$$

which is not bounded.

Hence again by Lemma 3.1, f is not bounded.

4. r-Norms on Γ

For every $r \in R^+$ an r-norm $|\,.:r\,|$ can be defined on Γ as:

$|\,.:r\,| : \Gamma \to R$ such that $|\,\alpha : r\,| = \Sigma\,|\,a_n\,|\,r^n$, where $\alpha = \Sigma a_n z^n$. It is easy to verify that $|\,.:r\,|$ is a norm on Γ (Iyer 1950). We denote Γ with this norm as $\Gamma(r)$. We denote by $\mathcal{B}_r$ the bornology on Γ consisting of the sets bounded in the sense of the norm $|\,.:r\,|$.

Theorem 4.1 — $\mathcal{B} = \bigcup_{r \in R^+} \mathcal{B}_r$

PROOF : Let $B \in \mathcal{B}$. Then there exists a constant k such that $\|\,\alpha\,\| \leqslant k$ for all $\alpha \in B$. Let now $\alpha = \Sigma a_n z^n \in B$. Then $|\,a_0\,| \leqslant k$, $|\,a_n\,|\ |\,z\,|^n \leqslant k^n\,|\,z\,|^n$, $n \geqslant 1$. Thus $\sum_0^\infty |\,a_n\,|\ |\,z\,|^n \leqslant k + \sum_1^\infty k^n\,|\,z\,|^n < \infty$ if $|\,z\,| < 1/k$. Hence if $0 < r < 1/k$ then $B \in \mathcal{B}_r$ and so $\mathcal{B} \subset \bigcup_{r \in R^+} \mathcal{B}_r$.

For the reverse inclusion let $B \in \mathcal{B}_r$. Then there exists a constant k such that $|\,\alpha : r\,| \leqslant k$ for all $\alpha = \Sigma a_n z^n \in B$.

i.e. $\quad \Sigma\,|\,a_n\,|\,r^n \leqslant k$

i.e. $\quad |\,a_0\,| \leqslant k,\ |\,a_n\,|^{1/n} \leqslant k^{1/n}/r,\ n \geqslant 1.$

i.e. $\quad \|\,\alpha\,\| \leqslant \text{Sup}\,\{k, k^{1/n}/r, n \geqslant 1\} < \infty.$

Thus $B \in \mathcal{B}$ and hence $\bigcup_{r \in R^+} \mathcal{B}_r \subset \mathcal{B}$.

Corollary 4.1 — (Γ, B) is the bornological inductive limit of the normed spaces $\{\Gamma(r)\}_{r \in R^+}$ where the inductive limit is defined in the usual way.

Lemma 4.1 — The following are equivalent for (Γ, B)

(a) $\alpha_n \xrightarrow{M} 0.$

(b) There exists a sequence $\{\lambda_n\}$ of positive real numbers tending to zero such that $\{\alpha_n/\lambda_n\}$ is bounded.

PROOF : (a) $\Rightarrow$ (b) is obviously true.

To prove (b) $\Rightarrow$ (a), let $\{\alpha_n\}$ be a sequence in Γ for which there exists a sequence $\{\lambda_n\}$ of positive real numbers tending to zero and a constant k such that $\|\,\alpha_n/\lambda_n\,\| \leqslant k$ for all n. Now there exists a positive number M such that $\lambda_n \leqslant M$ for all n. Further, we can choose for each $i = 1, 2, \ldots$ an n_i such that $\lambda_n < 1/i$ for all $n \geqslant n_i$. Let us define a sequence $\{\lambda_n^1\}$ as

$$\lambda_n^1 = M \text{ for all } n < n_1$$

$$= 1/i \text{ for all } n_i \leqslant n < n_{i+1},\ i = 1, 2, \ldots$$

Then $\{\lambda_n^1\}$ is a decreasing sequence of positive real numbers tending to zero and further

$$\lambda_n^1 \geqslant \lambda_n \text{ for all } n.$$

Hence
$$\| \alpha_n/\lambda_n^1 \| = \| \alpha_n\lambda_n/\lambda_n\lambda_n^1 \|$$

$$\leqslant A(\lambda_n/\lambda_n^1)\ \| \alpha_n/\lambda_n \|$$

$$\leqslant k. \quad \text{Therefore} \quad \alpha_n \xrightarrow{M} 0.$$

Theorem 4.2 — $\alpha_q \xrightarrow{M} 0$ in Γ if and only if $\alpha_q(z) \to 0$ uniformly in some finite circle.

PROOF : *Only if part* — Suppose $\alpha_q \xrightarrow{M} 0$ and $\alpha_q = \Sigma a_{q,n} z^n$. Then there exists a constant k and a sequence $\{\lambda_q\}$ in C, tending to zero such that $\| \alpha_q/\lambda_q \| \leqslant k$ for all q.

i.e.
$$\left| \frac{a_{q,0}}{\lambda_q} \right| \leqslant k \text{ and } \left| \frac{a_{q,n}}{\lambda_q} \right| \leqslant k^n, \quad n \geqslant 1.$$

i.e.
$$| a_{q,0} | \leqslant | \lambda_q | k \text{ and } | a_{q,n} | \leqslant | \lambda_q | k^n,\ n \geqslant 1.$$

If $z \in C$ such that $| z | \leqslant 1/2k$ then

$$| \alpha_q(z) | = | \sum_n a_{q,n} z^n | \leqslant \sum_n | a_{q,n} | \ | z |^n$$

$$\leqslant \sum_n | \lambda_q | k^n | z |^n + | \lambda_q | k$$

$$= | \lambda_q | \ (\sum_n k^n | z |^n + k)$$

$$\leqslant | \lambda_q | \left(\sum_{n=1}^{\infty} 1/2^n + k \right)$$

$$\leqslant | \lambda_q | (1 + k)$$

Hence $| \alpha_q(z) | \to 0$ uniformly for all z such that $| z | \leqslant \frac{1}{2k}$.

If part — Suppose there exists an $r \in R^+$ such that $\alpha_q(z) \to 0$ uniformly for all z such that $| z | \leqslant r$.

i.e. $$\sup_{|z| \leqslant r} |\alpha_q(z)| \to 0 \quad \text{as } q \to \infty.$$

Now $|\alpha_q(z)| \leqslant \sup_{|z| \leqslant r} |\alpha_q(z)|$ for all z such that $|z| \leqslant r$. Hence by using the theorem of Cauchy's estimates we get, $|a_{q,n}| r^n \leqslant \sup_{|z| \leqslant r} |\alpha_q(z)|$.

i.e. $$\left[\frac{|a_{q,n}|}{\sup_{|z| \leqslant r} |\alpha_q(z)|}\right]^{1/n} \leqslant 1/r.$$

Let $\lambda_q = \sup_{|z| \leqslant r} |\alpha_q(z)|$. Then $|a_{q,0}| = |\alpha_q(0)| \leqslant \lambda_q$ and $\lambda_q \to 0$ Hence

$$\| \alpha_q/\lambda_q \| = \sup_{n \geqslant 1} \left\{ \frac{|a_{q,0}|}{|\lambda_q|}, \frac{|a_{q,n}|}{|\lambda_q|} \right\}^{1/n}$$

$$\leqslant \max \{1, 1/r\} = A(1/r).$$

and hence, in view of Lemma 4.1 above, $\alpha_q^M \to 0$.

Theorem 4.3 — The bornological dual $\Gamma(r)^\times$ of $\Gamma(r)$ is the same as its topological dual $\overline{\Gamma(r)}$.

PROOF: The proof follows immediately from the fact that a linear functional on a normed linear space is continuous if and only if it is bounded.

On $\overline{\Gamma(r)}$ we now define a map:

$$| . : 1/2r | : \overline{\Gamma(r)} \to R$$

$$\alpha = \Sigma a_n z^n \mid\to |\alpha : 1/2r| = \sum_0^\infty |a_n| / 2^n r^n.$$

It is well-known that $\alpha = \Sigma a_n z^n \in \overline{\Gamma(r)}$ if and only if $\{ |a_n| / r^n\}$ is bounded (Iyer 1950). Consequently the function $| . : 1/2r |$ is well-defined and $\overline{\Gamma(r)}$ becomes a normed linear space relative to $| . : 1/2r |$.

Denote by $\bar{\mathscr{B}}_{1/2r}$ the canonical bornology of $\overline{\Gamma(r)}$ with this norm which we call the $1/2r$-norm.

5. THE SPACE $\bar{\Gamma}$

In this section we consider the set $\bar{\Gamma} = \{\beta = \Sigma\, b_n z^n \mid b_n \in C$ and $\{ |b_n|^{1/n}\}$ is bounded$\}$. A convex bornology $\bar{\mathscr{B}}$ can be defined on $\bar{\Gamma}$ with the help of a function $\| . \| : \bar{\Gamma} \to R$ defined in a similar fashion to that on Γ (see Iyer 1948). We note that $\bar{\mathscr{B}}$ when restricted to Γ gives $\mathscr{B}$. Moreover,

$$\Gamma = \bigcup_{r \in R^+} \Gamma(r) \text{ (algebraically)},$$

and as in the proof of Theorem 4.1 we have that $\bar{\mathscr{B}} = \bigcup_{r \in R^+} \bar{\mathscr{B}}_r$. Consequently $\bar{\Gamma}$ is the bornological inductive limit of the normed spaces $\bar{\Gamma}(r)$.

Theorem 5.1 — $(\bar{\Gamma}, \bar{\mathscr{B}})$ is M-complete.

PROOF : We first observe that Lemma 4.1 holds for $\bar{\Gamma}$ also. Let thus $\{\alpha_n\}$ be an M-Cauchy sequence in $\bar{\Gamma}$. Then there exists a sequence $\{\mu_{nm}\}$ of scalars, tending to zero, such that $\left\| \frac{\alpha_n - \alpha_m}{\mu_{nm}} \right\| \leqslant k$, where k is some fixed real positive number.

Now we choose a sequence $\{\lambda_{nm}\}$ of scalars such that $\lambda_{nm} \geqslant \mu_{nm}$ for all n, m and further such that $\lambda_{n_1 m_1} \leqslant \lambda_{n_2 m_2}$ whenever $n_1 \geqslant n_2$ and $m_1 \geqslant m_2$. For this, since $\mu_{nm} \to 0$, without loss we can assume that $\mu_{nm} < 1$ for all n, m. Now we set $n_1 = 1$, $m_1 = 1$ and choose (n_i, m_i) inductively such that $n_i > n_{i-1}$, $m_i > m_{i-1}$ and $\mu_{nm} < 1/i$ for $n \geqslant n_i$, $m \geqslant m_i$. Define $\{\lambda_{nm}\}$ as

$$\lambda_{nm} = \frac{1}{\min(i, j)} \quad \text{if } n_i \leqslant n < n_{i+1} \text{ and } m_j \leqslant m \leqslant n_{j+1}.$$

It is easily seen that $\{\lambda_{nm}\}$ is the required sequence. Moreover, $\lambda_{nm} \to 0$ and

$$\left\| \frac{\alpha_n - \alpha_m}{\lambda_{nm}} \right\| \leqslant \left\| \frac{\alpha_n - \alpha_m}{\mu_{nm}} \right\| \leqslant k$$

i.e.
$$\frac{|a_{n,0} - a_{m,0}|}{|\lambda_{nm}|} \leqslant k \text{ and } \frac{|a_{n,q} - a_{m,q}|}{|\lambda_{nm}|} \leqslant k \text{ for all } q \geqslant 1$$

i.e. $\{a_{n,0}\}$, and $\{a_{n,q}\}$, $q \geqslant 1$, are Cauchy sequences and hence there exist a_0, a_q, $q \geqslant 1$ in C such that $a_{n,0} \to a_0$ and $a_{n,q} \to a_q$ for all $q \geqslant 1$.

$$\text{Now} \quad \frac{|a_{n,0} - a_{p,0}|}{|\lambda_{n,n+1}|} < \frac{|a_{n,0} - a_{p,0}|}{|\lambda_{n,p}|} \quad \text{for all } p \geqslant n+1$$

and hence

$$\frac{|a_{n,0} - a_{p,0}|}{|\lambda_{n,n+1}|} \leqslant k \text{ for all } p \geqslant n+1.$$

Hence as $p \to \infty$, we get $\frac{|a_{n,0} - a_0|}{|\lambda_{n,n+1}|} \leqslant k$.

Similarly

$$\frac{|a_{n,q} - a_q|^{1/q}}{|\lambda_{n,n+1}|} \leqslant k \text{ for all } q \geqslant 1.$$

i. e. $$\left\| \frac{\alpha_n - \alpha}{\lambda_{n,n+1}} \right\| \leqslant k \text{ where } \alpha = \Sigma\, a_n z^n \text{ and } \lambda_{n,n+1} \to 0$$

Hence $\alpha_n \overset{M}{\to} \alpha$.

Now $$\begin{aligned} | a_n |^{1/n} &= | a_{q,n} - a_n - a_{q,n} |^{1/n} \\ &\leqslant | a_{q,n} - a_n |^{1/n} + | a_{q,n} |^{1/n} \\ &\leqslant | \lambda_{q,q+1} | \cdot k + | a_{q,n} |^{1/n} \end{aligned}$$

Hence $$\overline{\lim} | a_n |^{1/n} \leqslant \overline{\lim} | \lambda_{q,q+1} | k + \overline{\lim} | a_{q,n} |^{1/n}$$
$$\leqslant Mk + r_q < \infty \quad \text{where} \quad M = \sup_q | \lambda_{q,q+1} | < \infty$$

and $r_q = \overline{\lim} | a_{q,n} |^{1/n} < \infty$. Hence $\alpha \in \bar{\Gamma}$ and therefore $\bar{\Gamma}$ is M-complete.

Corollary 5.1 — $\bar{\Gamma}$ is complete.

PROOF : In view of Theorem 1 in Hogbe-Nlend (1971, p. 33) it is enough to show that $\bar{\mathscr{B}}$ is l^1-disced. For this we show that each $B_r \in \bar{\mathscr{B}}$ is l^1-disced. Let thus $\{\lambda_i\}$ be a sequence of scalars such that $\sum_{i=1}^{\infty} | \lambda_i | \leqslant 1$, and $\{\alpha_i\}$ be a sequence in B_r. Then

$$\begin{aligned} \| \alpha = \sum_1^\infty \lambda_i \alpha_i \| &= \sup \{ | \sum_1^\infty \lambda_i a_{i,0} | , \ | \sum_1^\infty \lambda_i a_{i,n} |^{1/n}, \ n \geqslant 1 \} \\ &\leqslant \sup \{ \Sigma | \lambda_i | \ | a_{i,0} | , \ (\Sigma | \lambda_i | \ | a_{i,n} |)^{1/n}, \ n \geqslant 1 \} \\ &\leqslant \sup \{ r \Sigma | \lambda_i | , \ r (\Sigma | \lambda_i |)^{1/n}, \ n \geqslant 1 \} \\ &\leqslant r. \end{aligned}$$

Hence B_r is l^1-disced and the assertion follows.

Theorem 5.2 — $(\Gamma, \mathscr{B})$ is not complete.

PROOF : It is enough to show that $(\Gamma, \mathscr{B})$ is not M-complete, (see Hogbe-Nlend 1971, p. 33). Consider the sequence

$$\alpha_n = \sum_{i=1}^{n} (\tfrac{1}{2})^i z^i, \quad n \geqslant 1.$$

Then $\left\{ \frac{\alpha_n - \alpha_m}{(\frac{1}{2})^m}, \ n \geqslant m \right\}$ is bounded in Γ. In other words, $\{\alpha_n\}$ is an M-Cauchy sequence in Γ and hence in $\bar{\Gamma}$. As $(\bar{\Gamma}, \bar{\mathscr{B}})$ is M-complete, the M-limit of $\{\alpha_n\}$ exists in $\bar{\Gamma}$. In fact the M-limit of $\{\alpha_n\}$ in $\bar{\Gamma}$ is $\alpha = \sum_1^\infty (\frac{1}{2})^i z^i$ as $\{(\alpha_n - \alpha)/(\frac{1}{2})^n\}$ is bounded in

$\bar{\Gamma}$, and $\alpha \notin \Gamma$. We now claim that the M-limit of $\{\alpha_n\}$ does not exist in Γ. For otherwise, let $\alpha_n \xrightarrow{M} \beta \in \Gamma$. Then $\alpha_n \xrightarrow{M} \beta$ in $\bar{\Gamma}$. Hence $\beta = \alpha$ as $\bar{\Gamma}$ is a separated bornological vector space. This contradicts the fact that $\alpha \notin \Gamma$.

Theorem 5.3 — $(\bar{\Gamma}, \bar{\mathscr{B}})$ is the M-completion of $(\Gamma, \mathscr{B})$. In other words, every $\alpha \in \bar{\Gamma}$ can be written as the M-limit of a sequence $\{\alpha_n\}$ in Γ.

PROOF: Let $\alpha = \Sigma\, c_n z^n \in \bar{\Gamma}$. Then there exists a number d such that $|c_n|^{1/n} < d$ for all $n \geqslant 1$. Now consider the sequence $\{\alpha_q = \sum_0^q c_n z^n\}$, $q = 1, 2, \ldots$ in Γ. Then

$$\left\| \frac{\alpha - \alpha_q}{(\frac{1}{2})^q} \right\| = \left\| \sum_{q+1}^{\infty} \frac{c_n z^n}{(\frac{1}{2})^q} \right\|$$
$$< 2d < \infty.$$

The following corollary is immediate from Theorems 5.1 and 5.3.

Corollary 5.2 — $(\bar{\Gamma}, \bar{\mathscr{B}})$ is the completion of $(\Gamma, \mathscr{B})$.

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