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ON THE (N, p_n) C_1 SUMMABILITY OF A SEQUENCE OF FOURIER COEFFICIENTS

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Mohanty and Nanda (1954) were the first to establish a result for the $(C, 1)$ summability of the sequence $\{nB_n(x)\}$. Varshney (1959) improved this result for the $(N, 1/(n+1))$ C_1 summability. Sharma (1969, 1970) and Lal (1971) generalised the above results in different directions. The present author has discussed the summability (N, p_n) C_1 of the sequence $\{nB_n(x)\}$ and has obtained a more general result than that of Sharma.

§1. Let Σa_n be a given infinite series with the sequence of partial sums $\{S_n\}$. Let $\{p_n\}$ be a sequence of constants, real or complex, such that

$$P_n = p_0 + p_1 + p_2 + \dots + p_n,$$

$$P_{-1} = p_{-1} = 0.$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} S_k \qquad \dots(1.1)$$

defines the sequence $\{t_n\}$ of the Nörlund means of the sequence $\{S_n\}$, generated by the sequence of constants $\{p_n\}$. The series $\Sigma\, a_n$, or the sequence $\{S_n\}$, is said to be summable by Nörlund means (Nörlund 1919, Woronoi 1932), or summable (N, p_n) to the limit S, if $\lim_{n\to\infty} t_n = S$.

The conditions of regularity of the (N, p_n) method defined by (1.1) are

$$\lim_{n} \frac{p_n}{P_n} = 0 \qquad \dots(1.2)$$

and

$$\sum_{k=0}^{n} |\, p_k \,| = O(\,|\, P_n \,|\,), \quad \text{as} \quad n \to \infty. \qquad \dots(1.3)$$

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If p_n is real and nonnegative, (1.3) is automatically satisfied and then (1.2) is the necessary and sufficient condition for the regularity of the method (N, p_n) (Hardy 1967). In the special case in which

$$p_n = \frac{1}{n+1}$$

and therefore

$$P_n \sim \log n, \quad \text{as} \quad n \to \infty,$$

t_n reduces to the familiar harmonic mean of $\{S_n\}$ and if it is denoted by t'_n, then the series Σa_n, or the sequence $\{S_n\}$, is said to be summable by harmonic means to the sum S, if $\lim_{n\to\infty} t'_n = S$ (Riesz 1924).

If the method of summability (N, p_n) is superimposed on Cesàro means of order one, another method of summability (N, p_n) C_1 is obtained. For $p_n = \frac{1}{n+1}$, this method reduces to the summability method $\left(N, \frac{1}{n+1}\right) C_1$.

§2. Let $f(x)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over an interval $(-\pi, \pi)$. Let the Fourier series of $f(x)$ be

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x). \qquad \ldots(2.1)$$

The series conjugate to (2.1) is

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(x). \qquad \ldots(2.2)$$

We write

$$\phi(t) = f(x+t) + f(x-t)$$

$$\psi(t) = f(x+t) - f(x-t) - l, \quad \text{where } l \text{ is a constant.}$$

$$P(1/t) = P_\tau, \quad \text{where } \tau = (1/t) \text{ is the integral part of } 1/t.$$

$$\Psi(t) = \int_0^t |\psi(u)|\, du.$$

§3. Mohanty and Nanda (1954) proved the following theorem.

Theorem A — If

$$\psi(t) = o(t/\log(1/t)), \text{ as } t \to 0. \qquad \ldots(3.1)$$

and

$$a_n = O(n^{-\rho}), \quad b_n = O(n^{-\rho}), \quad 0 < \rho < 1, \qquad \ldots(3.2)$$

then the sequence $\{nB_n(x)\}$ is summable $(C, 1)$ to the value l/π.

Varshney (1959) proved the following theorem :

Theorem B — If

$$\int_0^t |\psi(u)| \, du = o\,(t/\log(1/t)), \text{ as } t \to 0, \qquad \ldots(3.3)$$

then the sequence $\{nB_n(x)\}$ is summable $\left(N, \frac{1}{n+1}\right) C_1$ to the value l/π.

Sharma (1969) generalised the above theorem of Varshney in the following form:

Theorem C — If

$$\int_0^t |\psi(u)| \, du = o(t), \; t \to 0 \qquad \ldots(3.4)$$

and

$$\int_t^\delta \frac{|\psi(u)|}{u} \log(1/u) \, du = o(\log 1/t), \quad 0 < \delta < \pi \qquad \ldots(3.5)$$

then the sequence $\{nB_n(x)\}$ is summable $\left(N, \frac{1}{n+1}\right) C_1$ to the value l/π.

Replacing the sequence $\{1/(n+1)\}$ by a more general sequence $\{p_n\}$, Lal (1971) established the following result for the (N, p_n) C_1 summability of the sequence $\{nB_n(x)\}$.

Theorem D — If the sequence $\{p_n\}$ is such that $|P_n| \to \infty$, as $n \to \infty$,

$$R_n \equiv \sum_{k=0}^{n} |p_k| = O(|P_n|) \qquad \ldots(3.6)$$

$$\sum_{k=1}^{n} k \, |p_k - p_{k+1}| = O(|P_n|), \quad \text{as } n \to \infty, \qquad \ldots(3.7)$$

and

$$\int_{1/n}^{\delta} \frac{|\psi(u)|}{u} R(1/u) \, du = o(R_n), \quad \text{as } n \to \infty, \qquad \ldots(3.8)$$

where $R(1/u)$ stands for $R_{[1/u]}$, $0 < \delta < \pi$, then the sequence $\{nB_n(x)\}$ is summable (N, p_n) C_1 to the value l/π.

Theorem B of Varshney was again generalised by Sharma (1970) in the following form:

Theorem E — Let $\{p_n\}$ be a monotonic non-increasing sequence of real positive constants such that

$$\left.\begin{array}{ll}\text{(i)} & P_n = p_0 + p_1 + p_2 + \ldots + p_n \to \infty, \quad \text{as} \quad n \to \infty \\ \text{(ii)} & \log n = O(P_n), \quad \text{as } n \to \infty\end{array}\right\} \qquad \ldots(3.9)$$

$$\text{(iii)} \quad \int_0^t |\psi(u)|\, du = o(t/P_\tau), \quad \text{as} \quad t \to 0, \text{ where } \tau = [1/t]. \qquad \ldots(3.10)$$

Then the sequence $\{nB_n(x)\}$ is summable (N, p_n) C_1 to the value l/π.

§4. We prove the following theorem:

Theorem — Let $p(u)$ be a monotonic decreasing and strictly positive for $u \geqslant 0$. Let $p_n = p(n)$ and

$$P(u) = \int_0^u p(x)\, dx, \text{ such that } P(u) \to \infty, \text{ as } u \to \infty. \qquad \ldots(4.1)$$

Let $\alpha(t)$ be a positive non-decreasing function of t. If

$$\Psi(t) \equiv \int_0^t |\psi(u)|\, du = o(t/\alpha(1/t)), \quad \text{as} \quad t \to +0 \qquad \ldots(4.2)$$

then a sufficient condition that the sequence $\{nB_n(x)\}$ be summable (N, p_n) C_1 to the value l/π, is that

$$\int_1^u \frac{P(t)}{t\alpha(t)}\, dt = O(P(u)), \quad \text{as} \quad u \to \infty. \qquad \ldots(4.3)$$

§5. We require the following lemmas to prove our theorem:

Lemma 1 (McFadden 1942) — If $\{p_n\}$ is nonnegative and non-increasing, then for $0 \leqslant a < b \leqslant \infty$, $0 \leqslant t \leqslant \pi$ and any n

$$\left| \sum_{k=a}^{b} p_k e^{i(n-k)t} \right| \leqslant AP_\tau.$$

Lemma 2 — If $0 \leqslant t \leqslant 1/n$, then

$$|Q_n(t)| = \left| \frac{1}{\pi P_n} \sum_{k=1}^{n} p_{n-k} \left(\frac{\sin kt}{kt^2} - \frac{\cos kt}{t} \right) \right| = O(n).$$

PROOF: $$| Q_n(t) | = O\left(\frac{1}{P_n} \sum_{k=1}^{n} p_{n-k}(k^2 t) \right)$$

$$= O\left(\frac{n}{P_n} \sum_{k=1}^{n} p_{n-k} \right) = O(n)$$

Lemma 3 (Dwivedi 1971) — For $0 < t \leqslant \pi$

$$| Q_n(t) | = O\left(\frac{P(1/t)}{tP_n} \right).$$

§6. *Proof of the theorem* — If we denote the $(C, 1)$ transform of the sequence $\{nB_n(x)\}$ by t_n, we have, after Mohanty and Nanda (1954), that

$$t_n - \frac{l}{\pi} = \frac{1}{n} \sum_{k=1}^{n} rB_r(x) - \frac{l}{\pi}$$

$$= \frac{1}{\pi} \int_0^{\pi} \psi(t) \left\{ \frac{1}{4n} \frac{\sin nt}{\sin^2 t/2} - \tfrac{1}{2} \frac{\cos nt}{\tan t/2} \right\} dt$$

$$+ \frac{1}{2\pi} \int_0^{\pi} \psi(t) \sin nt \, dt + o(1)$$

$$= \frac{1}{\pi} \int_0^{\pi} \psi(t) \left[\frac{\sin nt}{nt^2} - \frac{\cos nt}{t} \right] dt + o(1),$$

as the second integral becomes $o(1)$, by Riemann-Lebesgue theorem.

Since the method of summability under consideration is regular, we have to show that under the conditions of the theorem

$$I \equiv \int_0^{\pi} \frac{\psi(t)}{\pi P_n} \sum_{k=1}^{n} p_{n-k} \left[\frac{\sin kt}{kt^2} - \frac{\cos kt}{t} \right] dt$$

$$= o(1), \quad \text{as} \quad n \to \infty.$$

Let us write

$$Q_n(t) \equiv \frac{1}{\pi P_n} \sum_{k=1}^{n} p_{n-k} \left[\frac{\sin kt}{kt^2} - \frac{\cos kt}{t} \right].$$

Therefore,

$$I = \int_0^\pi \psi(t)\, Q_n(t)\, dt$$

$$= \Big[\int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi} \Big] \psi(t)\, Q_n(t)\, dt$$

$$= I_1 + I_2 + I_3, \text{ say, where } 0 < \delta < \pi.$$

Now

$$I_1 = \int_0^{1/n} \psi(t)\, Q_n(t)\, dt$$

$$= O\Big[n \int_0^{1/n} |\psi(t)|\, dt\Big], \text{ by Lemma 2.}$$

$$= O\Big[n\, o\Big(\frac{1}{n\alpha(n)}\Big)\Big], \quad \text{by condition (4.2)}$$

$$= o\Big(\frac{1}{\alpha(n)}\Big) = o(1).$$

$$I_2 = \int_{1/n}^{\delta} \psi(t)\, Q_n(t)\, dt$$

$$= O\Big(\frac{1}{P_n}\Big) \int_{1/n}^{\delta} |\psi(t)|\, \frac{P(1/t)}{t}\, dt$$

$$= O\Big(\frac{1}{P_n}\Big) \Big[\Psi(t)\, \frac{P(1/t)}{t}\Big]_{1/n}^{\delta} + O\Big(\frac{1}{P_n}\Big) \int_{1/n}^{\delta} \Psi(t)\, d\Big\{\frac{P(1/t)}{t}\Big\}$$

$$= O\Big(\frac{1}{P_n}\Big) + o\Big(\frac{1}{\alpha(n)}\Big) + O\Big(\frac{1}{P_n}\Big) \int_{1/n}^{\delta} \Psi(t)\, d\Big\{\frac{P(1/t)\, \alpha(1/t)}{t\alpha(1/t)}\Big\}$$

$$= O\Big(\frac{1}{P_n}\Big) + o\Big(\frac{1}{\alpha(n)}\Big) + O\Big(\frac{1}{P_n}\Big) \int_{1/n}^{\delta} o\Big(\frac{t}{\alpha(1/t)}\Big) \frac{P(1/t)}{t\alpha(1/t)}\, d\alpha(1/t)$$

$$+ O\Big(\frac{1}{P_n}\Big) \int_{1/n}^{\delta} o\Big(\frac{t}{\alpha(1/t)}\Big)\, d\Big\{\frac{P(1/t)}{t\alpha(1/t)}\Big\}\, \alpha(1/t)$$

(*equations continued on p.* 880)

$$= O\left(\frac{1}{P_n}\right) + o\left(\frac{1}{\alpha(n)}\right) + o\,(1) \int_{1/n}^{\delta} \frac{d\alpha(1/t)}{\{\alpha(1/t)\}^2} + o\left(\frac{1}{P_n}\right) \int_{1/n}^{\delta} td \left\{ \frac{P(1/t)}{t\alpha(1/t)} \right\}$$

$$= O\left(\frac{1}{P_n}\right) + o\left(\frac{1}{\alpha(n)}\right) + o(1) \left[- \frac{1}{\alpha(1/t)} \right]_{1/n}^{\delta}$$

$$+ o\left(\frac{1}{P_n}\right) \left\{ \left[\frac{tP(1/t)}{t\alpha(1/t)} \right]_{1/n}^{\delta} - \int_{1/n}^{\delta} 1 \cdot \frac{P(1/t)}{t\alpha(1/t)}\, dt \right\}$$

$$= O\left(\frac{1}{P_n}\right) + o\left(\frac{1}{\alpha(n)}\right) + o\left(\frac{1}{P_n}\right) \int_{1/n}^{\delta} \frac{P(1/t)}{t\alpha(1/t)}\, dt + o(1)$$

$$= O\left(\frac{1}{P_n}\right) + o\left(\frac{1}{\alpha(n)}\right) + o\left(\frac{1}{P_n}\right) . O\,(P_n), \qquad \text{by condition (4.3)}$$

$$= o(1).$$

Since the method of summation is regular, we have by Riemann-Lebesgue theorem

$$I_3 = \int_{\delta}^{\pi} \psi(t)\, Q_n(t)\, dt = o(1), \quad \text{as} \quad n \to \infty.$$

Thus the proof of the theorem is complete.

§7. *Remark* : Taking $p_n = p(n) = \frac{1}{n+1}$ [and hence $P_n = P(n) \sim \log n$] in our theorem we get Theorem B as a corollary of our theorem.

If we take $P(n) = \alpha(n)$ in our theorem then Theorem E becomes a particular case of our theorem.

Condition (4.3) is a necessary condition for the summability $(N, p_n)\, C_1$ of the sequence $\{nB_n(x)\}$ if (4.1), (4.2) and the condition

$$\frac{1}{P_n} \int_{1/n}^{\delta} |\, \psi(t)\, |\ \frac{P(1/t)}{t}\, dt = o\,(1),$$

as $n \to \infty$, holds.

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