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# ON THE DEGREE OF APPROXIMATION OF FUNCTIONS BY CERTAIN NEW BERNSTEIN TYPE POLYNOMIALS

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The results of Bernstein (1912-13) and Popoviciu (1935) are extended for Lebesgue integrable functions in $L_1$-norm by the newly defined polynomials

$$A_n^\alpha(f, x) = (n+1) \sum_{k=0}^{n} \left( \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\, dt \right) p_{n,k}(x; \alpha),$$

where

$$p_{n,k}(x; \alpha) = \binom{n}{k} \frac{x(x+k\alpha)^{k-1}(1-x+(n-k)\alpha)^{n-k}}{(1+n\alpha)^n}.$$

## 1. INTRODUCTION AND RESULTS

If $f(x)$ is a function defined on [0,1], the Bernstein polynomials $B_n^f(x)$ of $f$ is

$$B_n^f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x) \qquad \ldots(1.1)$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}. \qquad \ldots(1.2)$$

Bernstein (1912-13) proved that if $f(x)$ is continuous in the closed interval [0,1], then

$$B_n^f(x) \to f(x), \qquad \ldots(1.3)$$

uniformly as $n \to \infty$. This yields a simple constructive proof of Weierstrass's approximation theorem.

A more precise version of this result due to Popoviciu (1935) states that

$$|B_n^f(x) - f(x)| \leqslant \tfrac{5}{4}\, w_f(n^{-1/2}) \qquad \ldots(1.4)$$

where $w_f$ is the uniform modulus of continuity of $f$ defined by

$$w_f(h) = \max \{ \mid f(x) - f(y) \mid : x, y \in [0, 1], \mid x - y \mid \leqslant h\}.$$

A slight modification of Bernstein polynomials due to Kantorovič (1930) makes it possible to approximate Lebesgue integrable functions in the $L_1$-norm by the modified polynomials

$$P_n^f(x) = (n+1) \sum_{k=0}^{n} \left( \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\, dt \right) p_{n,k}(x) \quad \ldots(1.5)$$

where $p_{n,k}(x)$ is as defined in (1.2)

By Abel's formula (see Jensen 1902)

$$(x + y + n\alpha)^n = \sum_{k=0}^{n} \binom{n}{k} x(x + k\alpha)^{k-1} (y + (n-k)\alpha)^{n-k}. \quad \ldots(1.6)$$

If we put $y = 1 - x$ (see Chenney and Sharma 1964)

$$(1 + n\alpha)^n = \sum_{k=0}^{n} \binom{n}{k} x(x + k\alpha)^{k-1} (1 - x + (n-k)\alpha)^{n-k},$$

or

$$1 = \sum_{k=0}^{n} \binom{n}{k} \frac{x(x + k\alpha)^{k-1} (1 - x + (n-k)\alpha)^{n-k}.}{(1 + n\alpha)^n} \quad \ldots(1.7)$$

Thus, defining

$$p_{n,k}(x; \alpha) = \binom{n}{k} \frac{x(x + k\alpha)^{k-1} (1 - x + (n-k)\alpha)^{n-k}}{(1 + n\alpha)^n} \quad \ldots(1.8)$$

from (1.7) we have

$$\sum_{k=0}^{n} p_{n,k}(x; \alpha) = 1. \quad \ldots(1.9)$$

Now we define a polynomial analogous to (1.5) as

$$A_n^\alpha(f, x) = (n+1) \sum_{k=0}^{n} \left( \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\, dt \right) p_{n,k}(x; \alpha) \quad \ldots(1.10)$$

where $p_{n,k}(x; \alpha)$ is as defined in (1.8), and moreover, when $\alpha = 0$, (1.8) and (1.10) reduce to (1.2) and (1.5), respectively.

The functions (see Chenney and Sharma 1964)

$$S(\nu, n, x, y) = \sum_{k=0}^{n} \binom{n}{k} (x + k\alpha)^{k+\nu-1} (y + (n - k)\alpha)^{n-k}$$

satisfy the reduction formula

$$S(\nu, n, x, y) = x\, S(\nu - 1, n, x, y) + n\alpha S(\nu, n - 1, x + \alpha, y)$$

such that

$$xS(o, n, x, y) = (x + y + n\alpha)^n. \qquad \text{...(1.11)}$$

By repeated use of reduction formula and (1.11), we may show that

$$S(1, n, x, y) = \sum_{k=0}^{n} \binom{n}{k} k!\, \alpha^k (x + y + n\alpha)^{n-k}$$

Replacing $k$ ! by $\int_0^\infty t^k e^{-t}\, dt$ and using Binomial theorem, we obtain

$$S(1, n, x, y) = \int_0^\infty e^{-t} (x + y + n\alpha + t\alpha)^n\, dt. \qquad \text{...(1.12)}$$

Similarly

$$S(2, n, x, y) = \sum_{k=0}^{n} (x + k\alpha) \binom{n}{k} k\alpha^k S(1, n - k, x + k\alpha, y)$$

reduces to

$$S(2, n, x, y) = \int_0^\infty e^{-t}\, dt \int_0^\infty e^{-s}\, ds\, \{x(x + y + n\alpha + t\alpha + s\alpha)^n$$
$$+ n\alpha^2 s (x + y + n\alpha + t\alpha + s\alpha)^{n-1}\}.$$

In this paper we shall prove the corresponding results of approximation due to Bernstein and Popoviciu for Lebesgue integrable functions in the $L_1$-norm by our polynomial (1.10) in terms of $L_1$ modulus of continuity.

$$w_f(h)_{L_1} = \sup_{|t| \leqslant h} \int_0^1 | f(x + t) - f(x) |\, dx.$$

In fact, we state our results as follows:

*Theorem* 1 — If $f(x)$ is a continuous Lebesgue integrable function on [0, 1], the relation

$$\lim_{n \to \infty} A_n^{\alpha}(f, x) = f(x)$$

uniformly on this interval.

*Theorem* 2 — Let $f(x)$ be a continuous Lebesgue integrable on [0, 1], then

$$| A_n^{\alpha}(f, x) - f(x) | \leqslant \tfrac{5}{4}\, w(n^{-1/2})\, L_1.$$

## 2. Proof of Theorems 1 and 2

To prove our results we need the following lemmas (see Anwar Habib and Umar 1980.)

*Lemma* 1 — For all $x$ in [0, 1]

$$\sum_{k=0}^{n} k p_{n,k}(x : \alpha) \leqslant \frac{nx}{1+\alpha}.$$

*Lemma* 2 — For all $x \in [0, 1]$

$$\sum_{k=0}^{n} k(k-1)\, p_{n,k}(x; \alpha)$$

$$\leqslant \left\{ n(n-1)\, x \left( \frac{x + 2\alpha}{(1+2\alpha)^2} + \frac{(n-2)\alpha^2}{(1+3\alpha)^2} \right) \right\}.$$

*Lemma* 3 — For all values of $x$ in [0, 1] we have

$$(n+1) \sum_{k=0}^{n} \left( \int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2\, dt \right) p_{n,k}(x; \alpha) \leqslant \frac{x(1-x)}{4}.$$

*Proof of Theorem* 1

For all $x \in [0, 1]$, we have

$$| A_n^{\alpha}(f, x) - f(x) |$$

$$= (n+1) \sum_{k=0}^{n} \left( \int_{k/(n+1)}^{(k+1)/(n+1)} | f(t) - f(x) |\, dt \right) p_{n,k}(x; \alpha)$$

(*equation continued on p.* 886)

$$= (n+1) \sum_{|t-x|<\delta} \left( \int_{k/(n+1)}^{(k+1)/(n+1)} | f(t) - f(x) | \, dt \right) p_{n,k}(x; \alpha)$$

$$+ (n+1) \sum_{|t-x| \geqslant \delta} \left( \int_{k/(n+1)}^{(k+1)/(n+1)} | f(t) - f(x) | \, dt \right) p_{n,k}(x; \alpha)$$

$$= I_1 + I_2, \text{ (say)}$$

where $$I_1 = (n+1) \sum_{|t-x|<\delta} \left( \int_{k/(n+1)}^{(k+1)/(n+1)} | f(t) - f(x) | \, dt \right) p_{n,k}(x; \alpha)$$

$$\leqslant \frac{\epsilon}{2} (n+1) \sum_{|t-x|<\delta} \left( \int_{k/(n+1)}^{(k+1)/(n+1)} dt \right) p_{n,k}(x; \alpha)$$

(by uniform continuity of $f$)

$$\leqslant \frac{\epsilon}{2} (n+1) \sum_{k=0}^{n} \left( \int_{k/(n+1)}^{(k+1)/(n+1)} dt \right) p_{n,k}(x; \alpha) = \frac{\epsilon}{2}. \qquad \ldots(2.1)$$

Now we evaluate $I_2$

$$I_2 = (n+1) \sum_{|t-x| \geqslant \delta} \left( \int_{k/(n+1)}^{(k+1)/(n+1)} | f(t) - f(x) | \, dt \right) p_{n,k}(x; \alpha),$$

$$\leqslant 2M (n+1) \sum_{|t-x| \geqslant \delta} \left( \int_{k/(n+1)}^{(k+1)/(n+1)} dt \right) p_{n,k}(x; \alpha),$$

(by boundedness of $f$)

$$\leqslant 2M(n+1) \delta^{-2} \sum_{k=0}^{n} \left( \int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2 \, dt \right) p_{n,k}(x; \alpha),$$

$$= 2M \delta^{-2} (n+1) \sum_{k=0}^{n} \left( \int_{k/(n+1)}^{(k+1)/(n+1)} (t-x)^2 \, dt \right) p_{n,k}(x; \alpha)$$

$$\leqslant 2M \delta^{-2} \left( \frac{1}{4n} \right)$$ (by Lemma 3 and the fact that $x(1-x) \leqslant \frac{1}{4}$ on $[0, 1]$).

$$= \frac{M}{2n\delta^2}$$

and hence

$$| A_n^{\alpha} (f, x) - f(x) | \leqslant I_1 + I_2$$

$$\leqslant \frac{\epsilon}{2} + \frac{M}{2n\delta^2}.$$

For sufficiently large value of $n$, $\frac{M}{n\delta^2} < \epsilon$ (independent of $x$) and consequently

$$| A_n^{\alpha} (f, x) - f(x) | < \epsilon.$$

which completes the proof of Theorem 1.

*Proof of Theorem* 2

For arbitrary $x_1$, $x_2$ in $[0, 1]$ and $\delta > 0$, we denote $\lambda = \lambda(x_1, x_2; \delta)$ the integers $[ | x_1 - x_2 | \delta^{-1}]$; the difference $\{f(x_1) - f(x_2)\}$ is then a sum of $(\lambda + 1)$ differences of $f(x)$ on intervals of length $< \delta$.

Thus it follows

$$| A_n^{\alpha} (f, x) - f(x) |$$

$$\leqslant (n + 1) \sum_{k=0}^{n} \left( \int_{k/(n+1)}^{(k+1)/(n+1)} | f(x) - f(t) | \, dt \right) p_{n,k}(x; \alpha)$$

$$\leqslant (n + 1) \, w(\delta)_{L_1} \sum_{k=0}^{n} \left( \int_{k/(n+1)}^{(k+1)/(n+1)} [1 + \lambda(x, t; \delta)] \, dt \right) p_{n,k}(x; \alpha)$$

(by hypothesis together with modulus of the continuity)

$$= (n + 1) \, w(\delta)_{L_1} \Bigg[ \sum_{k=0}^{n} \left( \int_{k/(n+1)}^{(k+1)/(n+1)} dt \right) p_{n,k}(x; \alpha)$$

$$+ \sum_{\lambda \geqslant 1} \left( \int_{k/(n+1)}^{(k+1)/(n+1)} \lambda(x, t; \delta) \, dt \right) p_{n,k}(x; \alpha) \Bigg]$$

$$= w(\delta)_{L_1} \Bigg[ 1 + (n + 1) \, \delta^{-1} \sum_{\lambda \geqslant 1} \left( \int_{k/(n+1)}^{(k+1)/(n+1)} | x - t | \, dt \right)$$

$$\times \, p_{n,k} (x; \alpha) \Bigg]$$

*(equation continued on p.* 888)

$$\leqslant w(\delta)_{L_1}\left[ 1 + (n+1)\delta^{-2} \sum_{k=0}^{n} \left( \int_{k/(n+1)}^{(k+1)/(n+1)} (x-t)^2\, dt \right) \times p_{n,k}(x;\alpha) \right].$$

Since $x(1-x) \leqslant \frac{1}{4}$ on [0, 1] and so by Lemma 3, we have

$$| A_n^{\alpha}(f, x) - f(x) | \leqslant w(\delta)_{L_1}\left[1 + \delta^{-2}\left(\frac{1}{4n}\right)\right].$$

For $\delta = n^{-1/2}$, we get our required result

$$| A_n^{\alpha}(f, x) - f(x) | \leqslant \tfrac{5}{4}\, w\, (n^{-1/2})_{L_1}.$$

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