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A GENERAL SHEFFER SET OF POLYNOMIALS RELEVANT TO THE THEORY OF HYPERBOLIC DIFFERENTIAL EQUATIONS

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A systematic use is made of the finite operator calculus developed by Rota *et al.* (1973) to study a general Sheffer set $X_n^{(\alpha)}(x)$ relative to the invertible shift-invariant operator $(I + D)^{-\alpha}$ and the delta operator $[I - (I + D)^{-2}]$. For this polynomial set the basic sequence is investigated and generating relations, recurrence relations, Rodrigues' type formula, etc. are obtained.

1. Introduction

The purpose of this paper is to study a general class of polynomials $X_n^{(\alpha)}(x)$ which form a Sheffer set with respect to the delta operator $Q = I - (I + D)^{-2}$ and the invertible shift-invariant operator $S = (I + D)^{-\alpha}$, where D is the differential operator d/dx and α is any real number. For special values of α, $X_n^{(\alpha)}(x)$ include the polynomials $\pi_n(x)$ and $P_n(x)$ which have applications in theory of hyperbolic differential equations (Courant and Hilbert 1937, Chap. 5). In fact $X_n^{(1)}(x) = n!\,\pi_n(x)$ and $X_n^{(3)}(x) = \dfrac{(2n+1)!}{2^{2n}n!} P_n(x)$. Moreover $X_n^{(\alpha+1)}(-x) = n!\, z_n(x)$, where $z_n(x)$ are the polynomials studied by Preiser (1962) in his investigation of the biorthogonal polynomials derivable from ordinary differential equations of third order. Earlier Spencer and Fano (1951) had used $z_n(x)$ in the problem of penetration and diffusion of X-rays. Thus, by a simple adjustment, our results on $X_n^{(\alpha)}(x)$ also yield a large number of new results for $z_n(x)$.

In this paper, we make a systematic use of the finite operator calculus developed by Rota *et al.* (1973) to investigate this set. We obtain some representations, identities, generating relations and recurrence relations for $X_n^{(\alpha)}(x)$. We also give properties of the basic sequence $\gamma_n(x)$.

Notations and terminology are the same as of Rota *et al.* (1973).

2. Basic Sequence

If $\gamma_n(x)$ denotes the basic sequence for the Sheffer set $X_n^{(\alpha)}(x)$, then, since the delta operator is $Q = I - (I + D)^{-2}$, a generating relation for $\gamma_n(x)$ is given by (Rota *et al.* 1973, Corollary 3 to Theorem 3)

$$\sum_{n\geqslant 0} \gamma_n(x) \frac{t^n}{n!} = \exp\,(xq^{-1}(t)) \qquad \ldots(2.1)$$

where $q^{-1}(t)$ is the formal power series inverse to $q(t)$. Here $q(t)$ is the indicator of the delta operator, that is $Q = q(D)$. Since $q(t) = 1 - (1 + t)^{-2}$ and so

$$q^{-1}(t) = (I + t)^{-1/2} - 1,$$

we have the generating relation

$$\sum_{n\geqslant 0} \gamma_n(x) \frac{t^n}{n!} = \exp\,[x\{(1 - t)^{-1/2} - 1\}]. \qquad \ldots(2.2)$$

Expanding right hand side of (2.2) and equating coefficients of t^n, one gets

$$\gamma_n(x) = \sum_{k\geqslant 0} \sum_{l=0}^{k} \frac{x^k}{k!} \binom{k}{l} (-1)^{k-l} \left(\frac{l}{2} + n - 1\right)_n. \qquad \ldots(2.3)$$

The substitution $t = \dfrac{u(2 + u)}{(1 + u)^2}$ in (2.2) gives

$$\sum_{n\geqslant 0} \frac{\gamma_n(x)}{n!} \left[\frac{u(2 + u)}{(1 + u)^2}\right]^n = e^{xu}. \qquad \ldots(2.4)$$

Expansion of (2.4) in powers of u yields

$$\sum_{n\geqslant 0} \frac{x^n u^n}{n!} = \sum_{n\geqslant 0} \gamma_n(x) \frac{u^n}{n!} \sum_{k=0}^{n} \binom{n}{k} u^k 2^{n-k} \sum_{l\geqslant 0} \frac{(2n + l - 1)_l\, u^l}{l!}$$

$$= \sum_{n\geqslant 0} \sum_{l\geqslant 0} \sum_{2k\leqslant n} \frac{\gamma_{n-k}(x)\, u^{n+l} 2^{n-2k} (2n + l - 2k - 1)_l}{k!\,(n - 2k)!\, l!}$$

$$= \sum_{n\geqslant 0} \sum_{l=0}^{n} \sum_{2k<n} \frac{\gamma_{n-k-l}(x)\, u^n 2^{n-2k-l} (2n - 2k - l - 1)_l}{k!\,(n - 2k - l)!\, l!}.$$

Comparison of coefficients of u^n on both the sides leads to

$$x^n = \sum_{2k\leqslant n} \sum_{l=0}^{n} \frac{n!\, 2^{n-2k-l} (2n - 2k - l - 1)_l}{k!\,(n - 2k - l)!\, l!} \gamma_{n-k-l}(x). \qquad \ldots(2.5)$$

Next, differentiating (2.2) w.r.t. x and equating coefficients of t^n, we get

$$D\gamma_n(x) = \sum_{k=1}^{n} \binom{n}{k} \frac{(2k)!}{2^{2k}k!} \gamma_{n-k}(x). \qquad \ldots(2.6)$$

Moreover, since the sequence $\gamma_n(x)$ is of binomial type, it satisfies the binomial identity

$$\gamma_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k(x)\, \gamma_{n-k}(y). \qquad \ldots(2.7)$$

3. Polynomials $X_n^{(\alpha)}(x)$ as a Sheffer set

In this section, we study the set $X_n^{(\alpha)}(x)$ as the Sheffer set relative to the invertible shift-invariant operator $S = (I + D)^{-\alpha}$, where α, is any real number and the delta operator is $Q = I - (I + D)^{-2}$. Indeed, the basic sequence is $\gamma_n(x)$. Since $X_n^{(\alpha)}(x)$ is a Sheffer set relative to S, we have (Rota *et al.* 1973, §5)

$$X_n^{(\alpha)}(x) = S^{-1}\gamma_n(x) = (I + D)^{\alpha}\, \gamma_n(x). \qquad \ldots(3.1)$$

From the definition of a Sheffer set

$$QX_n^{(\alpha)}(x) = nX_{n-1}^{(\alpha)}(x) \qquad \ldots(3.2)$$

and in general

$$Q^p X_n^{(\alpha)}(x) = (n)_p\, X_{n-p}^{(\alpha)}(x). \qquad \ldots(3.3)$$

The binomial theorem for Sheffer polynomials yields the fundamental identity

$$X_n^{(\alpha)}(x + y) = \sum_{k=0}^{n} \binom{n}{k} X_k^{(\alpha)}(x)\, \gamma_{n-k}(y). \qquad \ldots(3.4)$$

A generating relation for this Sheffer set is given by (Rota *et al.*1973, §5)

$$\sum_{n\geq 0} X_n^{(\alpha)}(x) \frac{t^n}{n!} = \frac{1}{s(q^{-1}(t))} \exp\,(xq^{-1}(t)) \qquad \ldots(3.5)$$

where $s(t) = (1 + t)^{-\alpha}$ is the indicator of S and $q^{-1}(t)$ is the same as in §2, so that $s(q^{-1}(t)) = (1 - t)^{\alpha/2}$. Thus the generating relation turns out to be

$$\sum_{n\geq 0} X_n^{(\alpha)}(x) \frac{t^n}{n!} = (1 - t)^{-\alpha/2} \exp\,[x\,\{(1 - t)^{-1/2} - 1\}]. \qquad \ldots(3.6)$$

Substitution $t = u(2 + u)/(1 + u)^{-2}$ in (3.6) gives

$$\sum_{n\geqslant 0} \frac{X_n^{(\alpha)}(x)}{n!} \left[\frac{u(2+u)}{(1+u)^2}\right]^n = (1+u)^\alpha e^{xu}. \quad \ldots(3.7)$$

Proceeding on the lines of McBride (1971) we arrive at the following relation

$$\sum_{n\geqslant 0} X_{n+k}^{(\alpha)}(x) \frac{t^n}{n!} = (1-t)^{-(\alpha/2)-k} \exp\left[x\left\{(1-t)^{-1/2} - 1\right\}\right]$$

$$\times X_k^{(\alpha)}\left(x(1-t)^{-1/2}\right). \quad \ldots(3.8)$$

Then, following a technique of Srivastava and Lavoie (1975), we obtain the bilateral generating relation (see also Singhal and Srivastava 1972, p. 755):

$$\sum_{n\geqslant 0} X_n^{(\alpha)}(x)\, \nu_n(y) \frac{t^n}{n!} = (1-t)^{-\alpha/2} \exp\left[x\left\{(1-t)^{-1/2} - 1\right\}\right]$$

$$\times F\left[x(1-t)^{-1/2}; \frac{yt}{1-t}\right] \quad \ldots(3.9)$$

where

$$F(x, t) = \sum_{n\geqslant 0} \lambda_n X_n^{(\alpha)}(x) \frac{t^n}{n!}$$

and

$$\nu_n(y) = \sum_{k=0}^{n} \binom{n}{k} \lambda_k y^k,$$

$\lambda_n \neq 0$ $(n = 0, 1, 2, \ldots)$ are arbitrary constants.

4. Representations for the Polynomials $X_n^{(\alpha)}(x)$

(a) From (3.6) we have

$$\sum_{n\geqslant 0} X_n^{(\alpha)}(x) \frac{t^n}{n!} = \sum_{k\geqslant 0} e^{-x} \frac{x^k}{k!} (1-t)^{-(\alpha+k)/2}$$

$$= \sum_{k\geqslant 0} \sum_{n\geqslant 0} \frac{e^{-x} x^k}{k!} \left(\frac{\alpha+k}{2} + n - 1\right)_n \frac{t^n}{n!}$$

which on equating coefficients of t^n, leads to

$$X_n^{(\alpha)}(x) = \sum_{k \geqslant 0} e^{-x} \frac{x^k}{k!} \left(\frac{\alpha + k}{2} + n - 1 \right)_n$$

or

$$X_n^{(\alpha)}(x) = \sum_{l \geqslant 0} \frac{x^l}{l!} \sum_{k=0}^{l} \binom{l}{k} (-1)^{l-k} \left(\frac{\alpha + k}{2} + n - 1 \right)_n. \qquad \ldots(4.1)$$

Since the inner series on the right-hand side of (4.1) represents the lth difference of a polynomial of degree n in α, which vanishes for $l > n$, we can write (4.1) as

$$X_n^{(\alpha)}(x) = \sum_{l=0}^{n} \frac{x^l}{l!} \underset{\alpha}{\Delta^l} (\tfrac{1}{2}\alpha + n - 1)_n$$

where

$$\underset{\alpha}{\Delta} f(x) = f(\alpha + 1) - f(\alpha)$$

and

$$\underset{\alpha}{\Delta} f(x) = \sum_{k=0}^{l} (-1)^{l-k} \binom{l}{k} f(\alpha + k).$$

Hence, we have the representation

$$X_n^{(\alpha)}(x) = \exp(x \underset{\alpha}{\Delta}) (\tfrac{1}{2}\alpha + n - 1)_n. \qquad \ldots(4.2)$$

(b) Using the standard arguments of complex analysis, we get from (3.6) the contour integral representation

$$X_n^{(\alpha)}(x) = \frac{n!}{2\pi i} \int_C t^{-n-1}(1 - t)^{-\alpha/2} \exp[x\{(1 - t)^{-1/2} - 1\}]\, dt \qquad \ldots(4.3)$$

where C is a simple closed curve around the origin and lying within $| t | < 1$.

If we put $s = x^2(1 + t)^{-1}$ in (4.3), we get another contour integral representation

$$X_n^{(\alpha)}(x) = \frac{n!}{2\pi i} e^{-x} x^{2-\alpha} \int_{C_1} \frac{s^{(2n-2+\alpha)/2} e^{s^{1/2}}}{(s - x^2)^{n-1}}\, ds \qquad \ldots(4.4)$$

where C_1 is the circle $| s - x^2 | = \gamma$ with centre at x^2 and small radius γ. Evidently C_1 may be any small closed contour encircling $s = x^2$.

(c) Contour integral representations (4.3) and (4.4) immediately lead to the following Rodrigues' type formulae

$$X_n^{(\alpha)}(x) = \left[\frac{\partial^n}{\partial t^n}\left\{(1-t)^{-\alpha/2}\exp\left[x\{(1-t)^{-1/2}-1\}\right]\right\}\right]_{t=0} \quad \ldots(4.5)$$

and

$$X_n^{(\alpha)}(x) = e^{-x}x^{2-\alpha}\left[\frac{\partial^n}{\partial s^n}\{s^{(2n-2+\alpha)/2}e^{s^{1/2}}\}\right]_{s=x^2} \quad \ldots(4.6)$$

Moreover (4.6) can be rewritten as

$$X_n^{(\alpha)}(x) = x^{2-\alpha}e^{-x}\left[\frac{d}{d(x^2)}\right]^n (x^{2n-2+\alpha}e^x). \quad \ldots(4.7)$$

For $\alpha = 1$, (4.7) gives the important relation [Courant and Hilbert 1937, Chap. 5, (38)] [see also Erdélyi *et al.* 1955, §19.7, (65)]

$$\pi_n(x) = \frac{xe^{-x}}{n!}\left[\frac{d}{dx^2}\right]^n (x^{2n-1}e^x).$$

Substitution $\alpha = 3$ in (4.7) yields

$$\frac{(2n+1)!}{2^{2n}n!}P_n(x) = e^{-x}x^{-1}\left[\frac{d}{dx^2}\right]^n (x^{2n+1}e^x)$$

which on using Legendre's duplication formula

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\,\Gamma(z+\tfrac{1}{2}),$$

leads to the relation [Courant and Hilbert 1937, Chap. 5, (38)] (see also Erdélyi *et al.* 1955, §19.7 (67))

$$P_n(x) = \frac{\pi^{1/2}e^{-x}}{2x\Gamma(n+\frac{3}{2})}\left[\frac{d}{dx^2}\right]^n (x^{2n+1}e^x).$$

5. Application of Operators and Some Recurrence Relations

We prove the following results:

(a) $$D^k X_n^{(\alpha)}(x) = \underset{\alpha}{\Delta^k} X_n^{(\alpha)}(x) = \sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}X_n^{(\alpha+l)}(x)$$

(b) $$X_n^{(\alpha)}(x) - X_n^{(\alpha-2)}(x) = nX_{n-1}^{(\alpha)}(x)$$

(c) $$DX_n^{(\alpha)}(x) = \sum_{k=1}^{n}\binom{n}{k}\frac{(2k)!}{2^{2k}k!}X_{n-k}^{(\alpha)}(x)$$

and

(d) $$2X^{\alpha}_{n+1}(x) = (2n + \alpha + x)\, X_n^{(\alpha)}(x) + xDX_n^{(\alpha)}(x).$$

Proof of (*a*) : We prove this result by induction. From (3.1) we get that

$$DX_n^{(\alpha)}(x) = X_n^{(\alpha+1)}(x) - X_n^{(\alpha)}(x) = \underset{\alpha}{\Delta} X_n^{(\alpha)}(x)$$

which is (a) for $k = 1$.

Next assume the result for $k = m$.

Now,

$$D^{m+1}X_n^{(\alpha)}(x) = \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l}\, X_n^{(\alpha+l+1)}(x) - X_n^{(\alpha+l)}(x)$$

$$= \sum_{l=0}^{m+1} \binom{m+1}{l} (-1)^{m+1-l}\, X_n^{(\alpha+l)}(x)$$

and

$$\underset{\alpha}{\Delta}^{m+1} X_n^{(\alpha)}(x) = \underset{\alpha}{\Delta}(\underset{\alpha}{\Delta}^{m} X_n^{(\alpha)}(x))$$

$$= \underset{\alpha}{\Delta} D^m X_n^{(\alpha)}(x)$$

$$= D^m \underset{\alpha}{\Delta} X_n^{(\alpha)}(x)$$

$$= D^{m+1} X_n^{(\alpha)}(x).$$

Thus the result is established.

Proof of (*b*) : $X_n^{(\alpha)}(x) - X_n^{(\alpha-2)}(x)$

$$= (I + D)^{\alpha}\, \gamma_n(x) - (I + D)^{\alpha-2}\, \gamma_n(x) \qquad \text{(using (3.1))}$$

$$= (I + D)^{\alpha}\, Q\gamma_n(x) \qquad \text{(def. of } Q\text{)}$$

$$= QX_n^{(\alpha)}(x)$$

$$= nX_{n-1}^{(\alpha)}(x) \qquad \text{(using (3.2))}.$$

Proof of (*c*) : Differentiating (3.1) and then using (2.6), we obtain

$$DX_n^{(\alpha)}(x) = (I + D)^\alpha \sum_{k=0}^{n} \binom{n}{k} \frac{(2k)!}{2^{2k}k!} \gamma_{n-k}(x)$$

$$= \sum_{k=0}^{n} \binom{n}{k} \frac{(2k)!}{2^{2k}k!} X_{n-k}^{(\alpha)}(x)$$

which is the required result.

Proof of (d) : Let $w = (1 - t)^{-\alpha/2} \exp [x \{(1 - t)^{-1/2} - 1\}]$

so that $\sum_{n \geqslant 0} X_n^{(\alpha)}(x) \frac{t^n}{n!} = w.$

It is easy to see that w satisfies the partial differential equation

$$2(1 - t) \frac{\partial w}{\partial t} - x \frac{\partial w}{\partial x} - (x + \alpha) w = 0$$

which yields (d).

6. $X_n^{(\alpha)}(x)$ as a Cross-sequence

It is evidently seen that $P^{-\alpha} = (I + D)^\alpha$ form a one parameter group of shift-invariant operators and for the basic sequence $\gamma_n(x)$, the relation

$$X_n^{(\alpha)}(x) = P^{-\alpha}\gamma_n(x) \qquad ...(6.1)$$

holds; thus $X_n^{[\alpha]}(x)$ form a cross-sequence (Rota *et al.* 1973, §8)

$$X_n^{[\alpha]}(x) = X_n^{(\alpha)}(x) \qquad ...(6.2)$$

and we have the identity

$$X_n^{[\alpha+\beta]}(x + y) = \sum_{k=0}^{n} \binom{n}{k} X_k^{[\alpha]}(x)\, X_{n-k}^{[\beta]}(y) \qquad ...(6.3)$$

holds for all α and β and for any x and y. The substitution $\beta = -\alpha$ in (6.3) yields

$$\gamma_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} X_k^{[\alpha]}(x)\, X_{n-k}^{[-\alpha]}(y) \qquad ...(6.4)$$

which for $y = 0$ gives

$$\gamma_n(x) = \sum_{k=0}^{n} \binom{n}{k} (-\tfrac{1}{2}\alpha + n - k - 1)_{n-k} X_k^{[\alpha]}(x) \quad \ldots(6.5)$$

using (4.2).

Substitution $x = 0 = y$ in (6.3) yields

$$X_n^{[\alpha+\beta]}(0) = \sum_{k=0}^{n} \binom{n}{k} X_k^{[\alpha]}(0)\, X_{n-k}^{[\beta]}(0)$$

which on using (4.2) gives

$$(\alpha + \beta + n - 1)_n = \sum_{k=0}^{n} \binom{n}{k} (\alpha + k - 1)_k\, (\beta + n - k - 1)_{n-k}. \quad \ldots(6.6)$$

Remark : It is interesting to note that the sequence $\gamma_n(x)$ is a particular case of the Sheffer set $X_n^{(\alpha)}(x)$ of which it is a basic sequence. In fact, $X_n^{(0)}(x) = \gamma_n(x)$. Thus (2.7) follows from (3.4). Moreover, we also get results for $\gamma_n(x)$ on putting $\alpha = 0$ in the relations (3.6) to (3.6), (4.1) and (4.3) to (4.7).

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