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EFFECT OF HIGH INITIAL STRESS ON THE PROPAGATION OF STONELEY WAVES AT THE INTERFACE OF TWO ISOTROPIC ELASTIC INCOMPRESSIBLE MEDIA

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The problem of Stoneley waves propagation at the interface of two isotropic elastic incompressible media stressed initially by a homogeneous deformation is investigated. The effect of the initial stress on the range of existence of this type of waves in Neo-Hookean materials is shown.

INTRODUCTION

The problem of wave propagation in initially deformed elastic medium has attracted wide attention. Mention may be made of the works of Hays and Rivlin (1961), Flavin (1963), Green (1963), etc. The wave propagation at the interface of two semi-infinite elastic media was first studied by Stoneley (1924) who obtained the frequency equation. Scholte (1947), Koppe (1948) and Cagniard (1962) investigated the range of existence of this type of waves for different values of elastic constants. In this paper we examine the Stoneley wave propagation, when two elastic half-spaces are under initial homogeneous deformation. The media are considered to be incompressible Mooney materials. Green's formulation (Green and Zerna 1963) of plane wave propagation superposed on initial deformation have been used to obtain frequency equation. Equations for the range of existence of Stoneley waves in the prestressed media are obtained for a special type of material, namely Neo-Hookean solids. Numerical results are presented in a graph which exhibits the fact that the effect of initial stress is to change the range of existence of Stoneley waves.

FORMULATION

Let us consider two semi-infinite isotropic homogeneous incompressible elastic media which are in welded contact. The upper medium B_0 is $z \geqq 0$ and the lower medium $\bar{B}_0$ is $z \leqq 0$, $z = 0$ being the interface. The system $B_0 + \bar{B}_0$ is subjected to uniform homogeneous deformation and the deformed system is $B + \bar{B}$. We identify the Cartesian coordinates in $B + \bar{B}$ by (x, y, z) where

$$x = \lambda_1 x_0, \quad y = \lambda_2 y_0, \quad z = \lambda_3 z_0 \qquad ...(1)$$

where λ_1, λ_2, λ_3 are constant extension ratios along the axes.

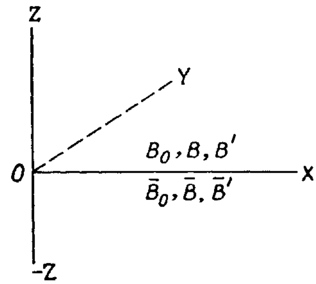

FIG. 1.

The fundamental metric in $B + \bar{B}$ following Green (1963) is $G_{ij} = G^{ij} = \delta_{ij}$, $i, j = 1, 2, 3$ and those of $B_0 + \bar{B}_0$ referred to the same coordinates is

$$g_{ij} = \begin{bmatrix} 1/\lambda_1^2 & 0 & 0 \\ 0 & 1/\lambda_2^2 & 0 \\ 0 & 0 & 1/\lambda_3^2 \end{bmatrix}, \quad g^{ij} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}.$$

The body is assumed to be incompressible so that the third strain invariant

$$I_3 = G/g = \lambda_1^2 \lambda_2^2 \lambda_3^2 = 1 \qquad \dots(2)$$

The three strain invariants for this homogeneous deformation are

$$\left.\begin{aligned} I_1 &= g^{ij}G_{ij} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\ I_2 &= g_{ij}G^{ij}I_3 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 \\ I_3 &= G/g = \lambda_1^2\lambda_2^2\lambda_3^2. \end{aligned}\right\} \qquad \dots(3)$$

The initial stress components are given by

$$\tau^{ij} = \phi g^{ij} + \psi B^{ij} + pG^{ij}$$

where $\quad \phi = \dfrac{2}{(I_3)^{1/2}} \dfrac{\partial W}{\partial I_1}, \quad \psi = \dfrac{2}{(I_3)^{1/2}} \dfrac{\partial W}{\partial I_2}$

and p is to be determined from boundary conditions and $B^{ij} = I_1 g^{ij} - g^{ir}g^{js}G_{rs}$ and W is the strain energy function which is a function of I_1, I_2 here.

Therefore

$$\left.\begin{aligned} \tau^{11} &= \phi\lambda_1^2 + \psi\lambda_1^2(\lambda_2^2 + \lambda_3^2) + p \\ \tau^{22} &= \phi\lambda_2^2 + \psi\lambda_2^2(\lambda_1^2 + \lambda_3^2) + p \\ \tau^{33} &= \phi\lambda_3^2 + \psi\lambda_3^2(\lambda_1^2 + \lambda_2^2) + p \\ \tau^{12} &= \tau^{21} = \tau^{23} = \tau^{32} = \tau^{31} = \tau^{13} = 0. \end{aligned}\right\} \qquad \dots(4)$$

The boundary condition is that the stress component τ^{33} tends to zero as $|z| \to \infty$. This is satisfied if p is chosen as

$$p = -\phi\lambda_3^2 - \psi\lambda_3^2(\lambda_1^2 + \lambda_2^2). \qquad \ldots(5)$$

For Stoneley waves let us now consider an infinitesimal deformation represented by the displacement field $(u, 0, \omega)$ superposed on $B + \bar{B}$. This transforms $B + \bar{B}$ to $B' + \bar{B}'$ with stresses $(\tau^{ij} + \epsilon\tau'^{ij})$ and the fundamental metric in $B' + \bar{B}'$ is $G_{ij} + \epsilon G'_{ij}$ where ϵ is very small so that its higher powers may be neglected. Then

$$G'^{ij} = -G^{ir}G^{js}G'_{rs} = -G'_{ij} = \begin{bmatrix} 2\dfrac{\partial u}{\partial x} & 0 & \dfrac{\partial u}{\partial z} + \dfrac{\partial \omega}{\partial x} \\ 0 & 0 & 0 \\ \dfrac{\partial u}{\partial z} + \dfrac{\partial \omega}{\partial x} & 0 & 2\dfrac{\partial \omega}{\partial z} \end{bmatrix} \qquad \ldots(6)$$

$$G' = GG^{ij}G'_{ij} = 2\left(\frac{\partial u}{\partial x} + \frac{\partial \omega}{\partial z}\right).$$

The strain invariants in $B' + \bar{B}'$ are $I_1 + \epsilon I_1$, $I_2 + \epsilon I'_2$, $I_3 + \epsilon I'_3$ where

$$\left.\begin{aligned} I'_1 &= g^{ij}G'_{ij} = 2\left(\lambda_1^2\frac{\partial u}{\partial x} + \lambda_3^2\frac{\partial \omega}{\partial z}\right), \quad I'_3 = \frac{G'}{g} = 2\left(\frac{\partial u}{\partial x} + \frac{\partial \omega}{\partial z}\right) \\ I'_2 &= g_{ij}(G'^{ij}I_3 - G^{ij}I'_3) = -2\left(\frac{1}{\lambda_1^2}\frac{\partial u}{\partial x} + \frac{1}{\lambda_3^2}\frac{\partial \omega}{\partial z}\right) \\ &\quad + I'_3(\lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2). \end{aligned}\right\} \qquad \ldots(7)$$

The incompressibility condition is

$$I'_3 = 0 \quad \text{i.e.} \quad \frac{\partial u}{\partial x} + \frac{\partial \omega}{\partial z} = 0. \qquad \ldots(8)$$

Let the media consist of Mooney materials with strain energy function as

$$W = c_1(I_1 - 3) + c_2(I_2 - 3) \quad \text{and} \quad \bar{W} = \bar{c}_1(I_1 - 3) + \bar{c}_2(I_2 - 3)$$

respectively.

The incremental stresses are given by

$$\tau'^{11} = a_{11}\frac{\partial u}{\partial x} + a_{13}\frac{\partial \omega}{\partial z} + p', \quad \tau'^{22} = a_{12}\frac{\partial u}{\partial x} + a_{23}\frac{\partial \omega}{\partial z} + p',$$

$$\tau'^{33} = a_{13}\frac{\partial u}{\partial x} + a_{33}\frac{\partial \omega}{\partial z} + p',$$

(equation continued on p. 922)

$$\left.\begin{aligned}
&\tau'^{13} = \tau'^{31} = a_{55}\left(\frac{\partial u}{\partial z} + \frac{\partial \omega}{\partial x}\right), \quad \tau'^{23} = \tau'^{32} = \tau'^{12} = \tau'^{21} = 0\\
&\bar{\tau}'^{11} = \bar{a}_{11}\frac{\partial \bar{u}}{\partial x} + \bar{a}_{13}\frac{\partial \bar{\omega}}{\partial z} + \bar{p}', \quad \bar{\tau}'^{22} = \bar{a}_{12}\frac{\partial \bar{u}}{\partial x} + \bar{a}_{23}\frac{\partial \bar{\omega}}{\partial z} + \bar{p}',\\
&\bar{\tau}'^{33} = \bar{a}_{13}\frac{\partial \bar{u}}{\partial x} + \bar{a}_{33}\frac{\partial \bar{\omega}}{\partial z} + \bar{p}'\\
&\bar{\tau}'^{13} = \bar{\tau}'^{31} = \bar{a}_{55}\left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{\omega}}{\partial x}\right), \quad \bar{\tau}'^{23} = \bar{\tau}'^{32} = \bar{\tau}'^{12} = \bar{\tau}'^{21} = 0
\end{aligned}\right\} \qquad \ldots(9)$$

where

$$\begin{aligned}
&a_{11} = a_{33} = -2p, \quad \bar{a}_{11} = \bar{a}_{33} = -2\bar{p}, \quad a_{13} = 2\lambda_1^2\lambda_3^2\psi, \quad \bar{a}_{13} = 2\lambda_1^2\lambda_3^2\bar{\psi}\\
&a_{12} = 2\lambda_1^2\lambda_2^2\psi, \quad \bar{a}_{12} = 2\lambda_1^2\lambda_2^2\bar{\psi}, \quad a_{23} = 2\lambda_1^2\lambda_3^2\bar{\psi}, \quad \bar{a}_{23} = 2\lambda_2^2\lambda_3^2\bar{\psi}\\
&a_{55} = -p - \lambda_1^2\lambda_3^2\psi, \quad \bar{a}_{55} = -\bar{p} - \lambda_2^2\lambda_3^2\bar{\psi}, \quad -p = \lambda_3^2\phi + \lambda_3^2(\lambda_1^2 + \lambda_2^2)\psi\\
&-\bar{p} = \lambda_3^2\bar{\phi} + \lambda_3^2(\lambda_1^2 + \lambda_2^2)\bar{\psi}, \quad \phi = 2c_1, \quad \bar{\phi} = 2\bar{c}_1, \quad \psi = 2c_2, \quad \bar{\psi} = 2\bar{c}_2.
\end{aligned} \qquad \ldots(10)$$

The equations motion in the two media are

$$\left.\begin{aligned}
&(\tau^{11} + a_{55})\frac{\partial^2 u}{\partial x^2} + a_{55}\frac{\partial^2 u}{\partial z^2} + \frac{\partial p'}{\partial x} = \rho\frac{\partial^2 u}{\partial t^2}\\
&(\tau^{11} + a_{55})\frac{\partial^2 \omega}{\partial x^2} + a_{55}\frac{\partial^2 \omega}{\partial z^2} + \frac{\partial p'}{\partial z} = \rho\frac{\partial^2 \omega}{\partial t^2}\\
&(\bar{\tau}^{11} + \bar{a}_{55})\frac{\partial^2 \bar{u}}{\partial x^2} + \bar{a}_{55}\frac{\partial^2 \bar{u}}{\partial z^2} + \frac{\partial \bar{p}'}{\partial x} = \bar{\rho}\frac{\partial^2 \bar{u}}{\partial t^2}\\
&(\bar{\tau}^{11} + \bar{a}_{55})\frac{\partial^2 \bar{\omega}}{\partial x^2} + \bar{a}_{55}\frac{\partial^2 \bar{\omega}}{\partial z^2} + \frac{\partial \bar{p}'}{\partial z} = \bar{\rho}\frac{\partial^2 \bar{\omega}}{\partial t^2}
\end{aligned}\right\} \qquad \ldots(11)$$

and incompressibility conditions are

$$\frac{\partial u}{\partial x} + \frac{\partial \omega}{\partial z} = 0 \quad \text{and} \quad \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{\omega}}{\partial z} = 0. \qquad \ldots(12)$$

The continuity conditions of displacements and stresses at $z = 0$ are

$$u = \bar{u}, \quad \omega = \bar{\omega}, \quad \tau'^{33} = \bar{\tau}'^{33}, \quad \tau'^{13} = \bar{\tau}'^{13}. \qquad \ldots(13)$$

In Stoneley Waves, the disturbance is produced in the neighbourhood of the interface and thus we take the solutions of (11) and (12) as

$$\left.\begin{aligned}
&p' = k(\rho c^2 - \tau^{11})Ae^{-kz+ik(x-ct)}, \quad \omega = (Ae^{-kz} + Be^{-k\nu z})e^{ik(x-ct)}\\
&u = -i(Ae^{-kz} + \nu Be^{-k\nu z})e^{ik(x-ct)}, \quad \nu^2 = 1 - \frac{\rho c^2 - \tau^{11}}{a_{55}} > 0
\end{aligned}\right\} \qquad \ldots(14)$$

in the upper medium, and in the lower medium the solution is taken in the form

$$\left.\begin{aligned} \bar{p}' &= k(\bar{\rho}c^2 - \bar{\tau}^{11})\,\bar{A}e^{kz+ik(x-ct)}, \\ \bar{\omega} &= (-\bar{A}e^{kz} + \bar{B}e^{k\bar{\nu}z})\,e^{ik(x-ct)} \\ \bar{u} &= -i(\bar{A}e^{kz} - \bar{\nu}\bar{B}e^{k\bar{\nu}z})\,e^{ik(x-ct)}, \quad \bar{\nu}^2 = 1 - \frac{\bar{\rho}c^2 - \bar{\tau}^{11}}{\bar{a}_{55}} > 0. \end{aligned}\right\} \quad \ldots(15)$$

Substituting (14), (15) in (9) and using (10), we get the stresses:

$$\left.\begin{aligned} \tau'^{33} &= [-k(2a_{55} + \tau^{11} - \rho c^2)\,Ae^{-kz} - 2a_{55}k\nu Be^{-k\nu z}]\,e^{ik(x-ct)} \\ \tau'^{13} &= ika_{55}\,[2Ae^{-kz} + (1 + \nu^2)\,Be^{-k\nu z}]\,e^{ik(x-ct)} \\ \bar{\tau}'^{33} &= [-k(2\bar{a}_{55} + \bar{\tau}^{11} - \bar{\rho}c^2)\,\bar{A}e^{kz} + 2\bar{a}_{55}k\bar{\nu}\bar{B}e^{k\bar{\nu}z}]\,e^{ik(x-ct)} \\ \bar{\tau}'^{13} &= ik\bar{a}_{55}\,[-2\bar{A}e^{kz} + (1 + \bar{\nu}^2)\,\bar{B}e^{k\bar{\nu}z}]\,e^{ik(x-ct)}. \end{aligned}\right\} \quad \ldots(16)$$

The displacement and stress continuity conditions (13) at $z = 0$ give

$$\left.\begin{aligned} & A - \bar{A} + \nu B + \bar{\nu}\bar{B} = 0 \\ & A + \bar{A} + B - \bar{B} = 0 \\ & (2a_{55} + \tau^{11} - \rho c^2)\,A + (2\bar{a}_{55} + \bar{\tau}^{11} - \bar{\rho}c^2)\,\bar{A} - 2a_{55}\nu B \\ & \qquad\qquad - 2\bar{\nu}\bar{a}_{55}\bar{B} = 0 \\ & 2a_{55}A + 2\bar{a}_{55}\bar{A} + a_{55}(1 + \nu^2)\,B - \bar{a}_{55}(1 + \bar{\nu}^2)\,\bar{B} = 0 \end{aligned}\right\} \quad \ldots(17)$$

From (17) we get the frequency equations as

$$\Delta = 0 \qquad \ldots(18)$$

where Δ is the determinant of coefficients of $A, \bar{A}, B, \bar{B}$ in eqns. (17).

If there were no initial deformation i.e. $\lambda_1 = \lambda_2 = \lambda_3 = 1$ then eqn. (18) reduces to the classical Stoneley wave frequency equation in two semi-infinite incompressible media. It is known that the shear wave velocity (Flavin 1963) in homogeneously deformed Mooney material media are $c_s = \left(\frac{\tau^{11} + a_{55}}{\rho}\right)^{1/2}$, $\bar{c}_s = \left(\frac{\bar{\tau}^{11} + \bar{a}_{55}}{\bar{\rho}}\right)^{1/2}$. From (14) and (15) it therefore follows that the Stoneley wave velocity is less than the shear wave velocities of the two media. The result is true in the classical case when there is no initial deformation.

Neo-Hookean Solids: Here $W = c_1(I_1 - 3)$, $\bar{W} = \bar{c}_1(I_1 - 3)$ $\therefore$ $\psi = \bar{\psi} = 0$, $\phi = 2c_1$, $\bar{\phi} = 2\bar{c}_1$. For comparison purpose with the classical case we write $\phi = \mu$ $\bar{\phi} = \bar{\mu}$ where $\mu \times \bar{\mu}$ are Lame's constants in the two media. Thus

$$\tau^{11} = (\lambda_1^2 - \lambda_3^2)\,\phi, \quad \bar{\tau}^{11} = (\lambda_1^2 - \lambda_3^2)\,\bar{\phi}, \quad a_{55} = \lambda_3^2\,\phi, \quad \bar{a}_{55} = \lambda_3^2\bar{\phi},$$

$$\beta^2 = \frac{\phi}{\rho} = \frac{\mu}{\rho}, \quad \bar{\beta}^2 = \frac{\bar{\phi}}{\bar{\rho}} = \frac{\bar{\mu}}{\bar{\rho}}, \quad \nu^2 = \frac{1}{\lambda_3^2}\left(\lambda_1^2 - \frac{c^2}{\beta^2}\right),$$

$$\bar{\nu}^2 = \frac{1}{\lambda_3^2}\left(\lambda_1^2 - \frac{c^2}{\bar{\beta}^2}\right).$$

The frequency equation is

$$\Delta = \begin{vmatrix} 1 & -1 & -\nu & -\bar{\nu} \\ 1 & 1 & 1 & 1 \\ -(1+\nu^2) & q(1+\nu^2) & 2\nu & 2q\bar{\nu} \\ 2 & 2q & -(1+\nu^2) & q(1+\bar{\nu}^2) \end{vmatrix} = 0 \qquad \ldots(19)$$

where $\quad q = \dfrac{\bar{\phi}}{\phi} = \dfrac{\bar{\mu}}{\mu}.$

To find the range of existence of Stoneley wave, we put $c = \bar{c}_s$ when $\bar{c}_s < c_s$ and $c = c_s$ when $c_s < \bar{c}_s$, since c is always less than c_s and $\bar{c}_s$, in (19) and we get two curves (A) and (B) in $(\rho/\bar{\rho},\ \mu/\bar{\mu})$ plane whose equations are

$$\left(\frac{\mu}{\bar{\mu}}\right)^2 \left\{\left(1 + \frac{\lambda_1^2}{\lambda_3^2} - \frac{\lambda_1^2}{\lambda_3^2}\frac{\bar{\beta}^2}{\beta^2}\right)^2 - 4\frac{\lambda_1}{\lambda_3}\sqrt{1 - (\bar{\beta}^2/\beta^2)}\right\}$$

$$-\left(1 - \frac{\lambda_1}{\lambda_3}\sqrt{1-(\bar{\beta}^2/\beta^2)}\right) - \frac{\mu}{\bar{\mu}}\left\{2\left(1 - \frac{\lambda_1^2}{\lambda_3^2} - \frac{\lambda_1^2}{\lambda_3^2}\frac{\bar{\beta}^2}{\beta^2}\right)\right.$$

$$\left. -\left(3 + \frac{\lambda_1^2}{\lambda_3^2}\right)\frac{\lambda_1}{\lambda_3}\sqrt{1-(\bar{\beta}^2/\beta^2)} - \frac{\lambda_1^2}{\lambda_3^2}\frac{\bar{\beta}^2}{\beta^2}\sqrt{1-(\bar{\beta}^2/\beta^2)}\right\} = 0 \quad \ldots(\mathrm{A})$$

$$\left(\frac{\mu}{\bar{\mu}}\right)^2\left(1 - \sqrt{1-(\beta^2/\bar{\beta}^2)}\right) + \left\{\left(1 + \frac{\lambda_1^2}{\lambda_3^2} - \frac{\lambda_1^2}{\lambda_3^2}(\beta^2/\bar{\beta}^2)\right)^2\right.$$

$$\left. -4\frac{\lambda_1}{\lambda_3}\sqrt{1-(\beta^2/\bar{\beta}^2)}\right\} - \frac{\mu}{\bar{\mu}}\left\{2\left(1 + \frac{\lambda_1^2}{\lambda_3^2} - \frac{\lambda_1^2}{\lambda_3^2}(\beta^2/\bar{\beta}^2)\right)\right.$$

$$\left. -\left(3 + \frac{\lambda_1^2}{\lambda_3^2}\right)\frac{\lambda_1}{\lambda_3}\sqrt{1-(\beta^2/\bar{\beta}^2)} + \frac{\lambda_1^2}{\lambda_3^2}\frac{\beta^2}{\bar{\beta}^2}\sqrt{1-(\beta^2/\bar{\beta}^2)}\right\} = 0. \quad \ldots(\mathrm{B})$$

The region of the $(\rho/\bar{\rho},\ \mu/\bar{\mu})$ plane lying between the curves (A) and (B) gives real Stoneley wave velocity c (Scholte 1947).

Conclusion

(1) If one medium is absent say $\bar{\rho} = 0$, then eqn. (18) gives the Rayleigh wave frequency equation in other half space. It may be verified that (Koppe 1948) the Stoneley wave velocity c is greater than the Rayleigh wave velociy c_R.

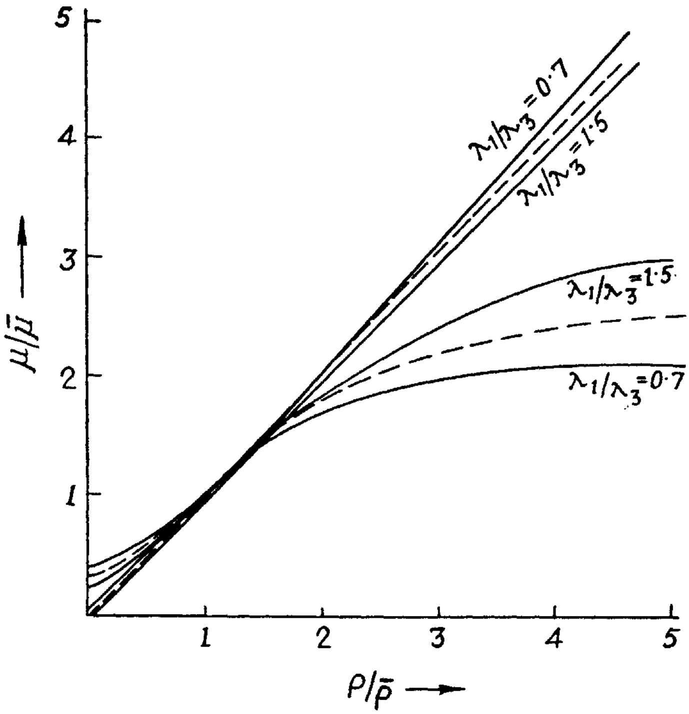

FIG. 2.

(2) The values of $\mu/\bar{\mu}$ for different values of $\rho/\bar{\rho}$ are calculated numerically from the two eqns. (A) and (B) with given values of λ_1^2/λ_3^2. The results are plotted in Fig. 2 to show the range of existence of Stoneley waves for different values of $\rho/\bar{\rho}$, $\mu/\bar{\mu}$. From eqns. (A), (B) we see that $\lambda_1/\lambda_3 = 1$ corresponds to the case of no initial stress in the body (Scholte 1947).

In Fig. 2 the dotted curves give the classical case. It is seen that if $\lambda_1/\lambda_3 > 1$ the range of existence becomes narrower, while if $\lambda_1/\lambda_3 < 1$ the range becomes wider. The effect of initial deformation for $\lambda_1/\lambda_3 = 0.7$, 1.5 are shown in Fig. 2.

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