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FIXED AND PERIODIC POINTS FOR SETS OF OPERATORS

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Some interesting results concerning periodic and fixed points of a sequence of operators have been presented in this note.

Let (X, d) be a metric space. Then a mapping $T : X \to X$ is said to be a contraction mapping if there exists a constant λ, $0 \leqslant \lambda < 1$ such that $d(Tx, Ty) \leqslant \lambda d(x, y)$ for all x, y in X. Banach's fixed point theorem states that a contraction mapping on a complete metric space has a unique fixed point.

Many authors have extended the theorem of Banach in different directions (Boyd and Wong 1969, Edelstein 1962, Kannan 1968, Pal and Maiti 1977).

We have deduced here a number of results in the case of sets of operators on a complete metric space which are presented in the following theorems.

Theorem 1 — Let T_i and T_j be two sets of operators mapping a complete metric space (X, d) into itself and let there exist positive intergers p and q such that for each pair of x, y in X at least one of the following is true:

(i) $d(x, T_i^p x) + d(y, T_j^q y) \leqslant \alpha d(x, y)$, $1 \leqslant \alpha < 2$

(ii) $d(x, T_i^p x) + d(y, T_j^q y)$

$$\leqslant \beta \{d(x, T_j^q y) + d(y, T_i^p x) + d(x, y)\}, \quad \tfrac{1}{2} \leqslant \beta < \tfrac{2}{3}$$

(iii) $d(x, T_i^p x) + d(y, T_j^q y) + d(T_i^p x, T_j^q y)$

$$\leqslant \gamma \{d(x, T_j^q y) + d(y, T_i^p x)\}, \quad 1 \leqslant \gamma < \tfrac{3}{2}$$

(iv) $d(T_i^p x, T_j^q y) \leqslant \delta \max \{d(x, y), \; d(x, T_i^p x), d(y, T_j^q y),$

$$\tfrac{1}{2}[d(x, T_j^q y) + d(y, T_i^p x)]\}, \quad 0 \leqslant \delta < 1.$$

Let a sequence $\{x_n\}$ in X be defined as follows:

$$x_{2n+1} = T^p_{2n+1}\, x_{2n}, \quad x_{2n+2} = T^q_{2n+2}\, x_{2n+1}, \quad n = 0, 1, 2, \ldots$$

Then T_i and T_j have a common periodic point u for

$$i = 1, 3, \ldots 2n+1, \ldots$$

$$j = 2, 4, \ldots 2n+2, \ldots$$

PROOF : Let $c_{2n} = d(x_{2n}, x_{2n+1})$. Suppose x_{2n}, x_{2n+1} satisfy (i). Then

$$d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) \leqslant \alpha d(x_{2n}, x_{2n+1}) \text{ i.e.}$$

(a) $c_{2n+1} \leqslant (\alpha - 1)\, c_{2n}$.

Similarly, if x_{2n}, x_{2n+1} satisfy (ii), (iii) or (iv) we get respectively,

(b) $c_{2n+1} \leqslant \dfrac{2\beta - 1}{1 - \beta}\, c_{2n}$

(c) $c_{2n+1} \leqslant \dfrac{\gamma - 1}{2 - \gamma}\, c_{2n}$

(d) $c_{2n+1} \leqslant \delta c_{2n}$.

Combining (a), (b), (c) and (d) we have

$$c_{2n+1} \leqslant \lambda c_{2n}$$

where $$\lambda = \max\left\{\alpha - 1, \frac{2\beta - 1}{1 - \beta}, \frac{\gamma - 1}{2 - \gamma}, \delta\right\} < 1.$$

Repeating the above argument with x_{2n-1}, x_{2n} we obtain

$$c_{2n} \leqslant \lambda\, c_{2n-1}.$$

Therefore $\{c_n\}$ converges to zero fast enough so that $\{x_n\}$ is Cauchy and hence converges to a point u in X.

We wish to show that $T^p_i\, u = T^q_j\, u = u$.

Suppose (i) holds. Then we have

$$d(u, T^p_i\, x) + d(x_{2n+1}, \ T^q_{2n+1}\, x_{2n+1}) \leqslant \alpha d(x_{2n+1}, u).$$

Proceeding to the limit $n \to \infty$ we get

$$u = T^p_i\, u.$$

Similar conclusion follows from (ii) to (iv).

Thus $u = T^p_i\, u$. Similarly one can show that $T^q_j\, u = u$.

Thus u is a common periodic point of T_i and T_j.

Remarks : If in addition to (i) to (iv) in Theorem 1, one has

(v) $d(u, T_i^p x) < d(x, u) + d(x, T_i^p x) \ \forall \ x$ in X, and

(vi) $d(u, T_j^q y < d(y, u) + d(y, T_j^q y) \ \forall \ y$ in X then u is a unique common fixed point for T_i and T_j.

PROOF : Suppose (v) is satisfied and assume $v \neq u$ is also a periodic point of T_i of period p. Then $d(u, v) < d(u, v) + d(u, T_i^p \ v) = d(u, v)$, a contradiction. Therefore $u = v$ and u is unique. Now $u = T_i^p u$ implies $T_i^p (T_i u) = T_i(T_i^p \ u) = T_i u$. But u is the unique periodic point of T_i and so $T_i u = u$. Similarly from (vi) $T_j u = u$. Moreover from (v) and (vi) it follows that u is the unique fixed point of T_i and T_j. This completes the proof.

Remark : The contractive definitions of Pal and Maiti (1977) and Ray (1978) follow as special cases of the contractive definitions in Theorem 1.

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