Indian J. pure appl. Math., 12(8) : 930–933, August 1981

COMMON FIXED POINTS OF SOME NONLINEAR MAPPINGS

R. N. MUKHERJEE

Mathematics Section, Department of Applied Sciences, Institute of Technology, Banaras Hindu University, Varanasi 221005

(*Received* 30 *October* 1980)

Some common fixed point theorems for commuting nonlinear mappings have been obtained in this note.

Jungck (1976) gave a Theorem for fixed points of commuting mappings which goes as follows:

Theorem — Let f be a continuous mapping of a complete metric space (X, d) into itself. Then f has a fixed point in X if and only if there exists $\alpha \in (0, 1)$ and mapping $g : X \rightarrow X$ which commutes with f and satisfies

$$g(X) \subset f(X) \text{ and } d(g(x), g(y)) \leqslant \alpha d(f(x), f(y)) \qquad (*)$$

for all $x, y \in X$.

Indeed f and g have a unique common fixed point if (*) holds. In the present paper we give some results on common fixed points of commuting mappings which satisfy more general condition than (*). First we state a lemma without proof from Jungck (1976) which goes as follows.

Lemma — Let $\{y_n\}$ be a sequence in a complete metric space (X, d). If there exists $\alpha \in (0, 1)$ such that

$$d(y_{n+1}, y_n) \leqslant \alpha d(y_n, y_{n-1}) \text{ for all } n,$$

then $\{y_n\}$ converges to a point in X.

Next we state our first result as follows.

Theorem 1 — Let f and g be mappings of a complete metric space into itself with f continuous. Let f and g commute with each other and $g(X) \subset f(X)$. Also let g satisfy the following condition:

$$d(g(x), g(y)) \leqslant a_1 d(g(x), f(x)) + a_2 d(g(y), f(y)) + a_3 d(g(x), f(y)) + a_4 d(g(y), f(x)) + a_5 d(f(x), f(y)) \qquad \ldots(1)$$

with $a_i \geqslant 0$ for all i and $a_1 + a_2 + a_3 + 2a_4 + a_5 < 1$ then f and g have a unique common fixed point in X.

PROOF : Let $x_0 \in X$, and let x_1 be such that $f(x_1) = g(x_0)$. In general choose x_n such that $f(x_n) = g(x_{n-1})$. We can do this since $g(X) \subset f(X)$.

From (1) we get

$$d(g(x_{n-1}), g(x_n)) \leqslant a_1 d(g(x_{n-1}), f(x_{n-1})) + a_2 d(g(x_n), f(x_n)) + a_3 d(g(x_{n-1}), f(x_n)) + a_4 d(g(x_n), f(x_{n-1})) + a_5 d(f(x_{n-1}), f(x_n))$$

i.e.,
$$d(f(x_n), f(x_{n+1})) \leqslant a_1 d(f(x_n), f(x_{n-1})) + a_2 d(f(x_{n+1}), f(x_n)) + a_3 d(f(x_n), f(x_n)) + a_4 d(f(x_{n+1}), f(x_{n-1})) + a_5 d(f(x_{n-1}), f(x_n))$$

i.e.
$$(1 - a_2 - a_3 - a_4 - a_5)\, d(f(x_n), f(x_{n+1})) \leqslant (a_1 + a_4)\, d(f(x_{n-1}), f(x_n))$$

i.e.
$$d(f(x_n), f(x_{n+1})) \leqslant \beta d(f(x_{n-1}), f(x_n)) \qquad \ldots(2)$$

where
$$\beta = \frac{a_1 + a_4}{1 - a_2 - a_3 - a_4 - a_5}.$$

It can be easily seen that $\beta < 1$ because of the conditions on $a_i (i = 1, 2, \ldots, 5)$.
Therefore invoking the lemma we get

$$t \in X \text{ such that } f(x_n) \to t.$$

Also

$$g(f(x_n)) = f(g(x_n)) = f(f(x_{n+1})) \to f(t)$$

since f is continuous.

Considerations which led to eqn. (2) also can reveal that

$$d(g(f(x_n)), g(g(t)) \to 0 \text{ as } n \to \infty.$$

Hence $g(t) = f(t)$.

Now we prove that $g(t)$ is a uniqne common fixed point of f and g.

$$d(g(g(t)), g(t)) \leqslant a_1 d(g(g(t)), f(g(t))) + a_2 d(g(t), f(t)) + a_3 d(g(g(t)), f(t)) + a_4 d(g(t), f(g(t))) + a_5 d(f(g(t)), f(t))$$

i.e.
$$(1 - a_3 - a_4 - a_5)\, d(g(g(t)), g(t)) \leqslant 0$$

$$\because a_3 + a_4 + a_5 < 1 \text{ hence } g(t) = g(g(t)).$$

We now have

$$f(g(t)) = g(f(t)) = g(g(t)) = g(t)$$

i.e., $g(t)$ is a common fixed point of f and g.

To see that f and g can have only one common fixed point, suppose $x = f(x) = g(x)$ and $y = f(y) = g(y)$, then

$$\begin{aligned} d(x, y) &= d(g(x), \; g(y)) \\ &\leqslant a_1 d(g(x), \; f(x)) + a_2 d(g(y), \; f(y)) \\ &\quad + a_3 d(g(x), \; f(y)) + a_4 d(g(y), \; f(x)) + a_5 d(f(x), \; f(y)) \end{aligned}$$

i.e. $$(1 - a_3 - a_4 - a_5)\, d(x, y) \leqslant 0$$

which implies that $d(x, y) = 0$.

If (X, d) happens to be a compact metric space then we have the following result which we state without proof.

Theorem 2 — Let f and g be two mappings of a compact metric space (X, d) into itself and f is continuous. Let f and g commute with each other and $g(X) \subset f(X)$. Also let g satisfy the following condition:

$$\begin{aligned} d(g(x), g(y)) \leqslant\; & a_1 d(g(x), f(x)) + a_2 d(g(y), \; f(y)) \\ & + a_3 d(g(x), \; f(y)) + a_4 d(g(y), \; f(x)) + a_5 d(f(x), \; f(y)) \end{aligned}$$

for $f(x) \neq f(y)$ with $a_i \geqslant 0$ for all i and $a_1 + a_2 + a_3 + 2a_4 + a_5 = 1$, then f and g have a unique common fixed point in X.

In an arbitrary topological space we give a theorem for common fixed point for a pair of commuting mappings which satisfy a condition of the type introduced by Chatterjee (1979a). As a corollary Theorem 1 of Chatterjee (1979a) follows.

Theorem 3 — Let f and g be two continuous mappings of a Hausdorff space X into itself and let f and g commute with each other and $g(X) \subset f(X)$. Let $F : X \times X \to R^+$ be a continuous mapping such that for each pair of $x, y \in X$ with $f(x) \neq f(y)$

$$F(g(x), g(y)) \leqslant \frac{\alpha F(f(y), g(y))\,[1 + F(f(x), g(x)]}{1 + F(f(x), f(y))} + \beta F(f(x), \; f(y))$$

with $\alpha, \beta \geqslant 0$ and $\alpha + \beta < 1$.

Let $x_0 \in X$, and x_1 be such that $f(x_1) = g(x_0)$.

In general choose x_n such that $f(x_n) = g(x_{n-1})$.

If the sequence $y_n = \{f(x_n)\}$ has a subsequence $\{y_{n_k}\}$ which converges to some $t \in X$ then $g(t)$ is a common fixed point of f and g.

PROOF : We know that $f(x_{n_k}) \to t$, from which it is also clear that $g(x_{n_k}) \to t$. Also since $f(g(x_{n_k})) = g(f(x_{n_k}))$, hence as $n_k \to \infty$ $f(g(x_{n_k})) \to f(t)$ and $g(f(x_{n_k})) \to g(t)$

since f and g are continuous $f(t) = g(t)$.

Now we show that $g(t)$ is the unique fixed point of f and g.

Now suppose $g(t) \neq g(g(t))$ then,

$$F(g(g(t)), \quad g(t))$$
$$\leqslant \frac{\alpha F(f(t), g(t))\,[1 + F(f(g(t)), g(g(t))]}{1 + F(f(g(t)), f(t))} + \beta F(f(g(t)), f(t)).$$

Since $\beta < 1$, therefore $F(g(g(t)), g(t)) < F(g(g(t)), g(t))$ which is a contradiction hence $g(t)$ is a fixed point of g. Since $f(g(t)) = g(f(t)) = g(g(t)) = g(t)$ hence $g(t)$ is also a fixed point of f.

Remark 1 : In case (X, d) is a metric space and F in Theorem 2 is replaced by d then the conditions of the above theorem yield a unique common fixed point of f and g.

Remark 2 : By taking f as the identity map we get Theorem 1 of Chatterjee (1979a) as a corollary of Theorem 3.

In case (X, d) is a compact metric space we can get a theorem which we state below without proof for a unique common fixed point in X for a pair of mappings f and g.

Theorem 4 — Let (X, d) be a compact metric space. Let f and g be a continuous, commutative pair of mappings taking X into itself such that $g(X) \subset f(X)$ and

$$d(g(x), g(y)) \leqslant \frac{\alpha d(f(y), g(y))\,[1 + d(f(x), g(x))]}{1 + d(f(x), f(y))} + \beta d(f(x), f(y))$$

with $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$ and $f(x) \neq f(y)$.

Then f and g have a unique common fixed point.

Remark 3 : In case f is taken as the identity map we get Theorem 1 of Chatterjee (1979b) as a corollary to Theorem 4.

REFERENCES

Chatterjee, H. (1979a). On generalization of Banach contraction principle. *Indian J. pure appl. Math.*, **10**, 400–403.

——————— (1979b). A fixed point theorem in metric spaces. *Indian J. pure appl. Math.*, **10**, 449–50.

Jungck, G. (1976). Commuting mappings and fixed points. *Am. Math. Monthly*, **83**, 261–63.