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A NOTE ON RADICALS IN LATTICE ORDERED ANTIFLEXIBLE RINGS

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In an earlier paper, we have studied the Prime, the (right) Jacobson and the nilpotent radical of a lattice ordered ring (Bhandari and Radhakrishna 1980a). For these radicals, in an arbitrary (nonassociative) ring, it is difficult to say anything more. However, the present note takes up the special case of an antiflexible ring. It is established that an lattice ordered antiflexible ring is right L-primitive if and only if it is left L-primitive.

A partially ordered (p.o.) ring (nonassociative) $(R, \leqslant)$ is a nonassociative ring R together with a partial order "$\leqslant$" satisfying

(i) $a \leqslant b$, $c \in R$ implies $a + c \leqslant b + c$

(ii) $a \leqslant b$, $0 \leqslant c$ implies $ac \leqslant bc$ and $ca \leqslant cb$.

A p.o. ring is said to be a lattice ordered (fully ordered) ring if $(R, +)$, the additive group of R is a lattice ordered (fully ordered) group. Since in any lattice ordered (l.o.) abelian group every element except zero is of infinite order we shall assume that characteristic R is 0.

Fuchs (1963) has studied l.o. (nonassociative) rings and proposed a study of various radicals in l.o. nonassociative rings. We have introduced (Bhandari and Radhakrishna 1980a) the concept of prime radical $P(R) = \cap\{P : P$ is an L-prime L-ideal of $R\}$, nilpotent radical $S(R) =$ sum of all nilpotent L-ideals of R and the (right) Jacobson radical $J_r(R) = \cap \{P : P$ is a right L-primitive L-ideals of $R\}$, where an L-ideal P is said to be

(i) L-prime if for any L-ideals K and I,

$$KI \equiv \{x \in R : |x| \leqslant |a| \cdot |b| \text{ for some } a \in K \text{ and } b \in I\} \subseteq P$$

implies either $K \supseteq P$ or $I \supseteq P$.

(ii) right L-primitive if there exist a maximal modular right L-ideal I with $I' = P$ where $I' = \{a \in R : \langle a \rangle \subseteq I\}$, $\langle a \rangle$ denotes the L-ideal generated by a.

For these radicals, in an arbitrary nonassociative ring, it is difficult to say anything more. However the present paper takes up the special case of an antiflexible ring, that is a (nonassociative) ring satisfying

$$(x, y, z) = (z, y, x) \qquad \ldots(1)$$

$$(x, x, x) = 0 \qquad \ldots(2)$$

where $(x, y, z) = (xy)\, z - x\,(yz)$. Antiflexible rings have been studied by Anderson and Outcalt (1968), Celik (1972), Rodabaugh (1965) and others. It is easy to verify that the following identity holds in any nonassociative ring:

$$(wx, y, z) - (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)\, z \qquad \ldots(3)$$

Antiflexible rings of characteristic 0 are power associative (Rodabaugh 1966). The following identities hold in any antiflexible ring (Anderson and Outcalt 1968).

$$(x, y, z) + (y, z, x) + (z, x, y) = 0 \qquad \ldots(4)$$

$$(w, [x, y], z) = 0 \qquad \ldots(5)$$

where $[x, y] \equiv xy - yx$.

By (1) the following two inequalities hold for all positive elements a, b, c of any l.o. antiflexible ring

$$(ab)\, c \leqslant a(bc) + (cb)\, a \qquad \ldots(6)$$

$$a(bc) \leqslant (ab)\, c + c(ba). \qquad \ldots(7)$$

An lattice ordered ring which is both a subring and a sublattice of a complete direct sum of f.o. rings is called an f-ring. An f-ring has following properties.

$$| ab | = | a | \cdot | b | . \qquad \ldots(8)$$

If $0 \leqslant c$, then for all $a, b \in R$,

$$\left.\begin{array}{l} (a \vee b)\, c = ac \vee bc,\ c(a \vee b) = ca \vee cb \\ \text{and} \\ (a \wedge b)\, c = ac \wedge bc \text{ and } c(a \wedge b) = ca \wedge cb \end{array}\right\} \qquad \ldots(9)$$

$$a^2 \geqslant 0 \text{ for every } a \in R. \qquad \ldots(10)$$

Let R be a l.o. antiflexible ring and let I be an L-ideal of R. Then similar to the case of right alternative rings (Bhandari and Radhakrishna 1980b) it is easy to verify that $I^2 \equiv \{x \in R : | x | \leqslant a^2, a \in I^+$ (the set of positive elements in I)$\}$ is an L-ideal and hence the following theorem holds.

Theorem 1 — In any l.o. antiflexible ring R, $S(R) \subseteq P(R)$. Moreover $S(R) = 0$ if and only if R is a subdirect sum of L-prime f.o. rings.

If R is a fully ordered (f.o.) antiflexible ring and $0 < a \in R$ such that $a^2 = 0$. Then for any $0 < x \in R$, $(xa)\, a - xa^2 = ax^2 - a(ax)$ or $(xa)\, a + a(ax) = 0$. Hence

$(xa)\,a = -a(ax) = 0$. Similarly using (4) we get $(ax)a = a(xa)$. If $ax \leqslant xa$, then $a(xa) = (ax)\,a \leqslant (xa)\,a = 0$ and if $xa < ax$, then $(ax)\,a = a(xa) \leqslant a(ax) = 0$. Thus $a(xa) = 0 = (ax)\,a$. Now using (6) and (7) we have $(ax)^2 \leqslant ((ax)a)\,x + x(a(ax))$. Also $(xa)^2 \leqslant x(a(xa)) + ((xa)a)\,x = 0$. Hence $I = \{a \in R : a^2 = 0\}$ is an L-ideal of R and the following theorem follows.

Theorem 2 — Let R be a f.o. antiflexible ring. If R is L-prime or if $S(R) = 0$ then R has no zero divisors.

Now let R be an antiflexible f-ring. If I is a right L-ideal of R then

$$(I : R) \equiv \{x \in R : Rx \subseteq I\}$$

is an L-ideal of R. For if $y, z \in R$, $x \in (I : R)$ then by (1) and (4),

$$y(xz) = -(y, x, z) + (yx)\,z = (x, z, y) + (x, y, z) + (yx)\,z \in I$$

and $y(zx) = -(y, z, x) + (yz)\,x = -(x, z, y) + (yz)\,x \in I$, and by (9) $|\,x\,| \in (I : R)$. Moreover if I is a maximal modular right L-ideal then $(I : R)$ is the largest L-ideal of R contained in I. Even more we have:

Lemma 3 — Any maximal modular right L-ideal I of an antiflexible f-ring R is an L-ideal of R.

PROOF : Let $(R, I, R) = \{(x, a, y) : x, y \in R, a \in I\}$ and let $P = (I : R)$. We first observe that $(R, I, R) \subseteq P$.

For if $x, y, z \in R$, $a \in I$ then by (1), (3) and (4),

$$\begin{aligned} z(x, a, y) &= (zx, a, y) - (z, xa, y) + (z, x, ay) - (z, x, a)\,y \\ &= -(a, y, zx) - (y, zx, a) - (z, xa, y) + (z, x, ay) - (z, x, a)\,y \\ &= -(a, y, zx) - (a, zx, y) - (z, xa, y) \\ &\quad + (ay, x, z) - (a, x, z)\,y, \end{aligned}$$

and by (5)
$$\begin{aligned} (z, xa, y) &= (z, ax, y) = -(ax, y, z) - (y, z, ax) \\ &= -(ax, y, z) - (ax, z, y). \end{aligned}$$

So $z(x, a, y) \in I$.

Now if I is not a two sided L-ideal, there exists $g \in R$, $0 < g$ such that $gI \nsubseteq I$. Let $K = \{x \in R : |\,x\,| \leqslant p + gy$ for some $p \in P^+, y \in I^+\}$. Since

$$(R, I, R) \equiv P \text{ and } (p + gy)\,r = pr + (g, y, r) + g(yr),$$

K is a right L-ideal of R containing P. As P is right L-primitive L-ideal, R/P is fully ordered and hence either $K \subseteq I$ or $I \subset K$. Since $gI \nsubseteq I$, $I = K$ and so $K = R$. Let e be a left identity modulo I. We can assume e to be positive, for e^2 is also a left identity modulo I. $ge \in K$. So there exist $p \in P^+$ and $b \in I^+$ such that $ge \leqslant p + gb$. Hence in the fully ordered ring R/P, $ge + P \leqslant gb + P$. By convexity of P, $e + P > b + P$ and so $ge + P \geqslant gb + P$. Thus $g(e - b) \in P$. The set $I_1 = \{x \in R : g\,x \in P\}$ is a right L-ideal of R containing P and so $I_1 \subseteq I$. Therefore $e \in I$, a contradiction.

If R is a right L-primitive f-ring then R is an L-simple fully ordered ring with a left identity, say f (Bhandari and Radhakrishna 1980a). If in addition R is antiflexible then R has no zero divisor and so f is unique. Moreover for any

$$x, y \in R,\ (x, f, y) = 0 \text{ and so } (f + (x - xf))\,y$$

$$= fy + xy - (xf)\,y = y.$$

By uniqueness of f, $x - xf = 0$. Hence f is the identity element of R. This proves:

Theorem 4 — An right L-primitive antiflexible f-ring is a fully ordered L-simple ring with identity and without zero divisors.

Since Theorem 4 is true, by symmetry for left L-primitive f-rings also, we have:

Corollary 5 — Let R be an antiflexible f-ring. Then R is left L-primitive if and only if R is right L-primitive.

Thus in any antiflexible f-ring R, $J_l(R) = J_r(R) = \bigcap \{P : P \text{ is a maximal } L\text{-ideal of } R \text{ such that } R/P \text{ has identity}\}$.

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