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D-REGULARITY FOR NEAR-RINGS

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The object of this paper is to study right D-regularity for near-rings. This helps us in producing an example of a non-hereditary radical class which gives rise to an F-radical class and thus supports the works of Scott (1972) and Saxena and Bhandari (1978, 1979).

1. INTRODUCTION

Various radicals of a near-ring are studied by Beidleman, Blackett, Laxton, Ramkotaiah and others. Scott (1972) has started an alternate general radical theoretic approach by defining a C-formation radical for a class C of near-rings. He has shown that the Baer lower radical gives rise to a C-formation radical class. In general a radical property need not give rise to a C-formation radical and vice versa. In sections 2 and 3 we generalize the concept "right D-regularity of rings" to near-ring. It helps us in constructing an example of a non-hereditary radical class which gives rise to a C-formation radical class. Characterization of $M_{r_0}(N)$ [the right D_0-regular radical of a distributively generated (d.g.) near-ring N] and $M_r(N)$ (the right D-regular radical of a near-ring N) are given and the relationship between $(M_{r_0}(N))_m$ and $M_{r_0}(N_m)$ are studied, where N_m represents the set of all $m \times m$ matrices over N.

Throughout this paper a near-ring means a left near-ring with $0 . x = 0$. A subset I of N is said to be an ideal of N if (i) $(I, +)$ is a normal subgroup of $(N, +)$, (ii) $(a + x) b\text{-}ab$ belongs to I for all $a, b \in N$, $x \in I$, and (iii) $ax \in I$ for all $a \in N$ and $x \in I$. If only (i) and (ii) are satisfied then I is called a right ideal. For other elementary definitions and properties we refer to Scott (1972). For a near-ring N and a natural number k, let N^k be the set $\{a_1 a_2 \ldots a_k : a_i \in N\}$. For a subset S of N, $\langle S \rangle$ ($\langle S \rangle_r$) will denote the ideal (right ideal) generated by the set S.

Let W be a homomorphically closed hereditary class, (called universal class) of near-rings. Similar to rings a class $P \subseteq W$ is called a radical class if (i) P is homomorphically closed, (ii) for each $N \in W$ there is a largest ideal $P(N)$ of N contained in P, and (iii) $N/P(N)$ has no nonzero ideal in P. A radical class P of near-rings is hereditary if and only if $P(N) \cap I \subseteq P(I)$ for all ideals I of N. P is called a stronger radical class if $P(I) \subseteq P(N)$ for all ideals I of N. If $P(N) \cap I = P(I)$ for

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all ideals I of N then P is called strongly hereditary. For associative rings hereditary and strongly hereditary properties are equivalent.

2. Right D_0-Regularity

Throughout this section, unless otherwise stated, N will represent a d.g. near-ring with a distributively generating set S. An element $x \in N$ is said to be right D_0-regular (r.D_0-r) if x is in the N-subgroup xN. A subset B of N is called r.D_0-r. if every element of B is right D_0-regular. If $\theta : N \to N'$ is a homomorphism and A is a r.D_0-r. subset of N then $A\theta$ is r.D_0-r. in N'. Moreover if θ is onto, Ker θ is r.D_0-r. in N and A' is a r.D_0-r. subset of N' then since N is a d.g. near-ring, $A'\theta^{-1}$ is r.D_0-r. in N. Thus for an ideal I of N, I and N/I are r.D_0-r. if and only if N is r.D_0-r. Let $M_{r_0}(N)$ be the sum of all r.D_0-r. ideals of N. Then $M_{r_0}(N)$ is the largest r.D_0-r. ideal of N satisfying $M_{r_0}(N/M_{r_0}(N)) = 0$. As an easy consequence we have the following.

Theorem 2.1 — The set of all right D_0-regular near-rings is a radical class.

An N-subgroup (right ideal) I of N is said to be large modular if there exists a nonzero element $x \in N$, $x \notin I$ such that $x\ N \subseteq I$, and $x \in J$ for all N-subgroups (right ideals) J with $J \supsetneqq I$. Using this we can give the following characterisation for $M_{r_0}(N)$.

Theorem 2.2 — For a d.g. near-ring N,

$$M_{r_0}(N) = \{x \in N : \langle x \rangle \text{ is r.}D_0\text{-r.}\}$$

$$= \bigcap \{I_\alpha : I_\alpha \text{ is the largest ideal contained in some large modular } N\text{-subgroup of } N\}.$$

Proof : The first part is obvious. For the second part, let $X = \bigcap \{I_\alpha : I_\alpha$ is the largest ideal contained in some large modular N-subgroup of $N\}$ and let $x \in N$ such that $x \notin X$. Then there exist a large modular N-subgroup J and an ideal I_α, the largest ideal contained in J such that $x \notin I_\alpha$. So there exist $y \in \langle x \rangle$ and $e \in N$ such that y, $e \notin J$, $eN \subseteq J$ and $e \in \bar{A}$, the N-subgroup generated by $\{y, J\}$. Since $\bar{A} \subseteq \langle y \rangle_r + J$, $e = c - m$ for some $c \in \langle y \rangle_r$ and $m \in J$. Clearly $c \notin J$. For any $z \in N$, $z = \sum_{i=1}^{k} s_i$, $\in S \cup -S$ and k some positive integer. So

$$cz = (e+m)\ z = \sum_{i=1}^{k} (e+m)\ s_i \in J.$$

Thus c and hence $\langle x \rangle$ cannot be r.D_0-r. in N. This contradiction proves that $M_{r_0}(N) \subseteq X$.

Conversely, let $x \notin M_{r_0}(N)$. Then there exists an element $y \in \langle x \rangle$ such that y is not r.D$_0$-r. in N. So $y \notin yN$. By Zorn's lemma there exists an N-subgroup J, maximal with respect to $y \notin J$ and $yN \subseteq J$. If I_α is the largest ideal of N contained in J, then $x \notin I_\alpha$. Thus $x \notin X$. Hence $X \subseteq M_{r_0}(N)$.

It is known that N_m, the set of all $m \times m$ matrices over N is a near-ring if and only if N is m-distributive (Ligh 1975). A near-ring N is said to be n-distributive if $(N^2, +)$ is abelian and $(a_1b_1 + a_2b_2 + \ldots + a_nb_n)r = a_1b_1r + a_2b_2r + \ldots + a_nb_nr$ for all a_i, b_i, r in N. For a d.g. near-ring N, the condition $(N^2, +)$ is abelian is equivalent to N being distributive. So the least we can restrict over N is to take N to be distributive. In that case N_m is also distributive and we have

Theorem 2.3 — For a distributive near-ring N and any natural number m, $(M_{r_0}(N))_m \subseteq M_{r_0}(N_m)$.

PROOF : We first note that if a-ax is r.D$_0$-r. in N, then a is r.D$_0$-r. in N. If x is r.D$_0$-r. in N, for convenience x' will denote an element in N satisfying $x = xx'$. Let I be a r.D$_0$-r. ideal of N and let $A = (a_{ij}) \in I_m$. We define m^2 matrices $A_{ij} = (b_{pq})$; $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, m$ as follows:

For $$i = 1, \; b_{pq} = \begin{cases} 0 & \text{if } q < j; \\ a_{pq} & \text{otherwise;} \end{cases}$$

and for $i = 2, 3, \ldots, m$,

$$b_{pq} = \begin{cases} 0 & \text{if } q < j \text{ or } q = j, p < i; \\ a_{pj}^{(i,j)} & \text{if } q < j, p \geqslant i; \\ a_{pq} & \text{otherwise;} \end{cases}$$

where

$$a_{pj}^{(1,j)} = a_{pj} \text{ and}$$

$$a_{pj}^{(i,j)} = a_{pj}^{(i-1,j)} - a_{pj}^{(i-1,j)} \left(a_{i-1,j}^{(i-1,j)}\right); \; i = 2, 3, \ldots, m.$$

Further let $\bar{A}_{ij} = (b_{pq})$ $i = 1, 2, \ldots, m$; $j = 1, \ldots, m$, be the m^2 matrices defined as follows:

For $$i = 1, \; b_{pq} = \begin{cases} a'_{1j} & \text{if } p = q = j; \\ 0 & \text{otherwise;} \end{cases}$$

and for $i = 2, 3, \ldots, m$,

$$b_{pq} = \begin{cases} a_{ij}^{(i,j)'} & \text{if } p = q = j; \\ 0 & \text{otherwise.} \end{cases}$$

Then it can be easily verified that $A_{ij} \in I_m$ for all $i, j = 1, 2, \ldots, m$; $A_{11} = A$;

$$A_{ij} = A_{i-1,j} - A_{i-1,j}\bar{A}_{i-1,j} \text{ for } i = 2, 3, \ldots, m; \quad j = 1, 2, \ldots, m;$$

$$A_{1j} = A_{m,j-1} - A_{m,j-1}\bar{A}_{m,j-1} \text{ for } j = 2, 3, \ldots, m; \text{ and}$$

$$A_{mm} - A_{mm}\bar{A}_{mm} = 0.$$

Thus A_{mm} is r.D_0-r. in N_m and hence A_{ij} is r.D_0-r. in N_m; $i, j = 1, \ldots, m$. Therefore A is r.D_0-r. and so I_m is a r.D_0-r. ideal of N_m. This completes the proof.

As easy consequences we have following two corollaries:

Corollary 2.4 — Let N be a distributive near-ring with a finite generating set S. Then $M_{r_0}(N) = N$ if and only if N is a ring having a right identity.

PROOF : Let $S = \{s_1, s_2, \ldots, s_n\}$ be a finite generating set of N and $M_{r_0}(N) = N$. Then the matrix $A = (b_{pq})$, $b_{p1} = s_p$ and $b_{pq} = 0$ for all $q > 1$, is r.D_0-r. in N_n. So there exists $b \in N$ such that $s_i b = s_i$ for all $i = 1, \ldots, n$. Since any $x \in N$ can be written as a finite sum of elements in $S \cup -S$ and N is distributive, $x = xb$ and hence b is a right identity of N.

The proof of the following corollary is similar to the proof of Corollary 2.4 and hence omitted.

Corollary 2.5 — If N is a distributive near-ring having an element which is not a left zero divisor then $M_{r_0}(N) = N$ if and only if N is a ring having a right identity.

It is easy to see that a d.g. near-ring N has a left identity if and only if there exists a r.D_0-r. element x which is not a left zero divisor (Ligh 1969).

3. Right *D*-Regularity

In the last section we restricted over d.g. near-rings due to the fact that the sum of all r.D_0-r. ideals may not be r.D_0-r. in case of arbitrary near-rings.

Hence we need to modify the definition as follows:

Definition 3.1 — An element x of a near-ring N is said to be right D-regular (r.D-r.) if $x \in \langle xN \rangle_r$. A nonempty subset S of N is said to be r.D-r. if every element of S is r.D-r.

If $\theta : N \to N'$ is an onto homomorphism, then it can be seen that

$$\langle aN \rangle_r \theta = \langle a\theta N' \rangle_r.$$

Hence homomorphic image of a r.D-r. subset is a r.D-r. subset. Moreover if $(a - z)$ is r.D-r. for $z \in \langle aN \rangle_r$ then for any $n \in N$, $(a - z)n = z_1 + an$, $z_1 \in \langle aN \rangle_r$.

Thus a is r.D-r. in N. As in section 2, for an ideal I of N, I and N/I are r.D-r. if and only if N is r.D-r. Let $M_r(N)$ denote the sum of all r.D-r. ideals of N. Then $M_r(N)$ is the largest ideal of N with $M_r(N/M_r(N)) = 0$ and we have the following:

Theorem 3.2 — The class of all r.D-r. near-rings is a radical class.

For $a, b, n \in N$, $(a + b)\, n = an + (-\, an) + ((a + b)\, n - an) + an = an + y$, $y \in \langle\, b\,\rangle_r$ and hence $\langle\, (a + b)\, N\,\rangle_r \subseteq \langle\, aN\,\rangle_r + \langle\, b\,\rangle_r$. If $b \in \langle\, aN\,\rangle_r$ then $\langle\, (a + b)N\,\rangle_r \subseteq \langle\, aN\,\rangle_r$. Using this fact the proof of the following theorem follows on the pattern given in the proof of Theorem 2.2 and hence omitted.

Theorem 3.3 — For any near-ring N,

$$M_r(N) = \{x : x \in N, \langle\, x\,\rangle \text{ is a r.D-r. ideal of } N\};$$

$$= \cap\, \{I_\alpha : I_\alpha \text{ is the largest ideal of } N \text{ contained in some large modular right ideal of } N\}.$$

We give below an example to show that the following two results for associative rings cannot be extended to near-rings.

Theorem 3.4 (Divinski 1958) — Let R be an associative ring with $M_r(R) = R$. R has an identity if and only if R has a non-zero divisor. If R has d.c.c. on left ideals then R has a right identity if and only if $R = M_r(R)$.

Theorem 3.5 (Divinski 1958) — Let R be a ring with d.c.c. on one-sided ideals. Then $R = M_r(R)$ if and only if R has a right identity. Further R has a right identity if and only if R has a non-right-zero divisor. If in addition R is commutative then $R = M_r(R)$ if and only if R has the identity. In fact the d.c.c. on one-sided ideals of $R/J(R)$ is sufficient in the last part.

Example 1 — Consider $N = \{0, a, b, c\}$ with addition and multiplication as given by following tables:

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	0	a	a
b	0	a	b	c
c	0	a	b	c

Then N is a near-ring (Clay 1968). Observe that $I = \{0, a\}$ is the only proper ideal of N and $J_0(N) = I$. Here $N = M_r(N)$ but N has no right identity.

If N is a subdirect sum of near-rings $\{N_i\}_{i \in I}$ then there exist a collection of ideals $\{I_i\}_{i \in I}$ of N such that $N/I_i \cong N_i$ and $\cap I_i = (0)$. So if $M_r(N_i) = 0$ for all $i \in I$ then $M_r(N) \subseteq \cap I_i = 0$. Thus we have proved the following theorem.

Theorem 3.6 — Let N be a subdirect sum of near-rings $\{N_i\}_{i \in I}$ with

$$M_r(N_i) = 0 \quad (M_{r_0}(N_i) = 0)$$

for all $i \in I$. Then $M_r(N) = 0 \quad (M_{r_0}(N) = 0)$.

4. Right *D*-regularity as an *F*-radical

Let C be a homomorphically closed class of near-rings and let

$$W = \{(I, N) : N \in C, \ I \text{ is an ideal of } N\}.$$

A class $P \subseteq W$ is called a C-formation radical (F-radical) class if

(1) $P = S_1P = QP = GP$;

(2) for each $N \in C$ there is an ideal $P(N)$ of N such that $(P(N), N) \in P$;

(3) $(N/P(N), N/P(N)) \in SP$, where for a class $M \subseteq W$;

(4) $S_1M = \{(J, N) : (I, N) \in M \text{ and } J \subseteq I\}$;

(5) $QM = \{(I\theta, N\theta) : (I, N) \in M, \ \theta \text{ is a homomorphism of } N\}$;

(6) $GM = \{(I/I \cap J, N/I \cap J) : ((I + J)/J, N/J) \in M$ for any ideals I, J of $N\}$; and

(7) $SM = \{(I, N) :$ there does not exist an ideal J of N with $J \subseteq I$ and $(J, N) \in M\}$.

Scott (1972) has shown that the Baer lower radical gives rise to a C-formation radical class. In general a radical property need not give rise to a C-formation radical class. In an unpublished note entitled 'Near-ring with minimal conditions on right N-subgroups' Scott has claimed that if P is a hereditary radical class then $\bar{P} = \{(I, N) : I \subseteq P(N)\}$ is a C-formation radical class. However there are non-hereditary radical classes which give rise to C-formation radical classes.

Theorem 4.1 — Radical class M_r of all right D-regular near-rings gives rise to a C-formation radical class.

Proof : It suffices to prove that $\bar{M}_r = G\bar{M}_r$. Let $((I + J)/J, N/J) \in \bar{M}_r$ and let β, γ be the natural mappings from $N/I \cap J$ to $(N/I \cap J)/(J/I \cap J)$ and from $(N/I \cap J)/(J/I \cap J)$ to N/J respectively. Let θ be the isomorphism from $I/I \cap J$ to $(I + J)/J$ given by $(x + I \cap J)\,\theta = x + J$. Then θ is the restriction of $\beta\gamma$ on

$I/I \cap J$. For any nonzero element $a + I \cap J$ of $I/I \cap J$, $(a + I \cap J)\theta = a + J \in I + J/J$. Since $I + J/J$ is r.D-r. in N/J, $(a + I \cap J)\theta \in F(a + J)$, where $F(a) = \langle aN \rangle_r$ for all $a \in N$. But $F(a + J) = F((a + I \cap J)\beta\gamma) = (F(a + I \cap J))\beta\gamma = (F(a + I \cap J))\theta$ as $F(a + I \cap J) \subseteq I/I \cap J$. Thus $(a + I \cap J)\theta = z\theta$ for some $z \in F(a + I \cap J)$ and hence $(a + I \cap J) = z$. Therefore $I/I \cap J$ is a r.D-r. ideal of $N/I \cap J$. This completes the proof.

We observe that $\bar{M}_r$ is a C-formation radical class which is not hereditary since M_r is not so even in case of associative rings (Divinski 1958).

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REFERENCES

Clay, J. R. (1968). Near-rings on groups of low order. *Math. Z.*, **104**, 364–71.

Divinski, N. J. (1958). *D*-regularity. *Proc. Am. math. Soc.*, **9**, 62–71.

Ligh, S. (1969). On d.g. near-rings. *Proc. Edinburgh Math. Soc.*, **16**, 239–42.

———(1975). A note on matrix near-ring. *J. Lond. Math. Soc.*, **11**, 383–84.

Saxena, P. K., and Bhandari, M. C. (1978). Lower formation radical for near-rings. *Kyungpook Math. J.*, **18**, 23–29.

———(1979). Lower and upper formation radical of near-rings. *Kyungpook Math. J.*, **19**, 205–11.

Scott, S. D. (1972). Formation radicals for near-rings. *Proc. Lond. math. Soc.*, **25**, 441–64.