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QUASI-PROXIMITY AND ASSOCIATED BITOPOLOGICAL SPACES

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A quasi-proximity on a non-empty set X is a proximity minus its symmetric property. Given a quasi-proximity on X there is another such on X called its conjugate, and X becomes a bitopological space. In this paper a characterisation of pairwise completely regular bitopological spaces has been established in terms of quasi-proximity.

§1. A non-empty set X on which are defined two topologies τ and $\mathcal{V}$ is called a bitopological space $(X, \tau, \mathcal{V})$. Kelly (1963) has studied consequences of introducing a quasi-metric, namely a distance function p on X satisfying the classical conditions of a metric minus the requirement of symmetry. Given a quasi-metric p on X one at once finds another quasi-metric q on X called conjugate of p by defining $q(x, y) = p(y, x)$ for $(x, y) \in X \times X$. This situation leads to a bitopological space (X, p, q) wherein Kelly (1963) has obtained some of the symmetry of the classical metric situation by proving generalisations of Urysohn's Lemma and Urysohn's metrization Theorem. The results thereby reveal that quasi-metrics are related to real-valued semi-continuous functions in much the same way as metrics are related to continuous functions. On the other hand one finds a proximity structure more general than uniform structure (see Naimpally and Warrack 1970) and it is wellknown that uniform structure is more general than a metric structure on X. In this chain of generalisations workers concerned have kept the symmetry property of the structure unchanged. Lane (1967) has dealt in good length a quasi-uniform structure i.e., a uniform structure minus its symmetry on X. We call a relation δ on the power set $\mathcal{P}(X)$ a quasi-proximity (Definition 1.1) on X and in this paper we study its consequences.

Definition 1.1 — A relation δ on $\mathcal{P}(X)$ is said to be a quasi-proximity on X iff it satisfies the axioms (Q 1 – Q 6) where

Q 1. $X \delta X$

Q 3. $A \subset B \lessdot C \subset D$ implies $A \lessdot D$

Q 4. $A \lessdot B_k (k = 1, 2, \ldots, n)$ implies $A \lessdot \bigcap_{k=1}^{n} B_k$

Q 5. $A_k \lessdot B (k = 1, 2, \ldots, n)$ implies $\bigcup_{k=1}^{n} A_k \lessdot B$

Q 6. $A \lessdot B$ implies existence of a subset C of X such that $A \lessdot C \lessdot B$.

A quasi-proximity $\lessdot$ on X induces a topology $\tau_{\lessdot}$ on X arising out of neighbourhood system $\mathcal{N}_x$ of $x \in X$ as described in Theorem 1.1.

Theorem 1.1 — For $x \in X$ let $\mathcal{N}_x = \{A \subset X : \{x\} \lessdot A\}$, then $\mathcal{N}_x$ is a neighbourhood system at x for a topology $\tau_{\lessdot}$.

PROOF: By routine verification the proof follows.

Theorem 1.2 — Given a quasi-proximity $\lessdot$ on X the relation $\lessdot^*$ on $\mathcal{P}(X)$ defined by $X \lessdot^* X$, and $A, B \in \mathcal{P}(X)$, $A \lessdot^* B$ iff $X \setminus B \lessdot X \setminus A$ is a quasi-proximity on X.

The proof is easy and left out.

Definition 1.2 — Quasi-proximities $\lessdot$ and $\lessdot^*$ are called conjugate to each other.

§2. By what has now been described so far $(X, \tau_{\lessdot}, \tau_{\lessdot^*})$ is a bitopological space. We have the following definitions.

Definition 2.1 (Equivalent to Definition 2.3 in Lane 1967) — In a bitopological space $(X, \tau, \mathcal{V})$, τ is said to be completely regular with respect to $\mathcal{V}$ if for each $G \in \tau$ and for each $x \in G$, there exists a τ-upper semi-continuous (τ-u.s.c.) and $\mathcal{V}$-lower semi-continuous ($\mathcal{V}$-l.s.c.) function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(X \setminus G) = 1$. $(X, \tau, \mathcal{V})$ is called pairwise completely regular if τ is completely regular with respect to $\mathcal{V}$, and $\mathcal{V}$ is completely regular with respect to τ.

Definition 2.2 — A bitopological space $(X, \tau, \mathcal{V})$ is said to be quasi-proximizable iff there is a quasi-proximity $\lessdot$ on X such that $\tau_{\lessdot} = \tau$ and $\tau_{\lessdot^*} = \mathcal{V}$.

Theorem 2.1 — A bitopological space $(X, \tau, \mathcal{V})$ is quasi-proximizable iff $(X, \tau, \mathcal{V})$ is pairwise completely regular.

The proof of Theorem 2.1 is accomplished by application of Lemmas 2.1 and 2.2 which we prove first.

Lemma 2.1 — Let $(X, \tau_{\lessdot})$ be a quasi-proximity space, the topology $\tau_{\lessdot}$ being induced by the given quasi-proximity $\lessdot$ on X. If $A \lessdot B$, then

$$A \ll \tau_{<}\text{-Int } B \tag{2.1.1}$$

and $$\tau_{<*}\text{-closure } A \ll B. \tag{2.1.2}$$

PROOF : Let $A \ll B$; so there is a subset C of X satisfying $A \ll C \ll B$. Now $\tau_{<} - \text{Int } B = \{x \in B : B \in \mathcal{N}_x\} = \{x \in B : \{x\} \ll B\}$, by Theorem 1.1.
If $c \in C$, then $c \in B$ and $\{c\} \subset C \ll B$ gives $\{c\} \ll B$, and hence $c \in \tau_{<}\text{-Int } B$. Thus we have $A \ll C \subset \tau_{<}\text{-Int } B$, and hence $A \ll \tau_{<}\text{-Int } B$, establishing (2.1.1). For proving (2.1.2), if possible, let $\tau_{<*}\text{-closure } A \not\subset C$; then there is

$$x \in (\tau_{<*}\text{-closure } A)\setminus C \text{ i.e. } x \in X\setminus C.$$

But $A \ll C$ gives $X\setminus C \ll^* X\setminus A$, showing thereby $X\setminus A \in \mathcal{N}_x$ (w.r.t. $\ll^*$) and $X\setminus A$ fails to meet A, a contradiction that $x \in \tau_{<*}\text{-closure } A$. So we have $\tau_{<*}\text{-closure } A \subset C$.

Further $C \ll B$ gives $\tau_{<*}\text{-closure } A \ll B$.

Lemma 2.2 — Let $(X, \tau_{<})$ be as in Lemma 2.1. If $A \ll B$, then there is a $\tau_{<}$-u.s.c. and $\tau_{<*}$-l.s.c. function $h : X \to [0, 1]$ such that

$$h(A) = 0 \text{ and } h(X\setminus B) = 1.$$

PROOF : The proof follows techniques as applied in classical Urysohn's Lemma. First applying Q.6 under Definition 1.1 twice we get subsets C and D such that $A \ll C \ll D \ll B$. Using Lemma 2.1 we have $A \subset \tau_{<*}\text{-closure } A \ll \tau_{<}\text{-Int } C \subset C \ll D \subset \tau_{<*}\text{-closure } D \ll \tau_{<} - \text{Int } B \subset B$. Put $\tau_{<*}\text{-closure } A = U_0$, $\tau_{<}\text{-Int } C = V_{1/2}$, $\tau_{<*}\text{-closure } D = U_{1/2}$, $\tau_{<}\text{-Int } B = V_1$. This gives

$$A \subset U_0 \ll V_{1/2} \ll U_{1/2} \ll V_1 \subset B,$$

and hence $A \subset U_0 \subset V_{1/2} \subset U_{1/2} \subset V_1 \subset B$. On repeating the construction for each pair of sets U_0, $V_{1/2}$ and $U_{1/2}$, V_1 we obtain $\tau_{<}$-open sets $V_{1/4}$, $V_{3/4}$ and $\tau_{<*}$-closed sets $U_{1/4}$, $U_{3/4}$ such that

$$A \subset U_0 \ll V_{1/4} \ll U_{1/4} \ll V_{1/2} \ll U_{1/2} \ll V_{3/4} \ll U_{3/4} \subset V_1 \subset B$$

and hence

$$A \subset U_0 \subset V_{1/4} \subset U_{1/4} \subset V_{1/2} \subset U_{1/2} \subset V_{3/4} \subset U_{3/4} \subset V_1 \subset B.$$

Continuing this process we have two families $\{U_s\}$ and $\{V_s\}$ of $\tau_{<*}$-closed sets and $\tau_{<}$-open sets respectively for $s = \dfrac{p}{2q}$ $(p = 1, 2, \ldots, 2^q - 1;\quad q = 1, 2, \ldots)$;

For any other dyadic rational s put $V_s = \phi$ for $s \leqslant 0$, $V_s = X$ for $s > 1$, and $U_s = \phi$ for $s < 0$, $U_s = X$ for $s \geqslant 1$. Then we have $V_r \subset V_s \subset U_s \subset U_t$ for $r \leqslant s \leqslant t$ and $U_s \subset V_t$ for $s < t$. Now define $h : X \to [0, 1]$ by the rule

$$h(x) = \inf\{t : x \in V_t\},\ x \in X.$$

It is not difficult to show that $h(x) = \inf\{t : x \in U_t\}$, $x \in X$; Clearly then $0 \leqslant h(x) \leqslant 1$ for $x \in X$; Further $h(x) = 0$ if $x \in U_0$, and hence $h(A) = 0$; and $h(x) = 1$ if $x \in X \setminus V_1$, and hence $h(X \setminus B) = 1$; Further using the sets U_s and V_s it is easy to show that h is $\tau_{\prec}$-u.s.c. and $\tau_{\prec^*}$-l.s.c. function; and proof is now complete.

Proof of Theorem 2.1 — Suppose $(X, \tau, \mathcal{V})$ is pairwise completely regular. Define a relation $\prec$ on $\mathcal{P}(X)$ by the rule: for A, $B \in \mathcal{P}(X)$, $A \prec B$ iff there is a τ-l.s.c. and $\mathcal{V}$-u.s.c. function $f : X \to [0, 1]$ such that $f(A) = 1$ and $f(X \setminus B) = 0$. We now verify that $\prec$ is a quasi-proximity on X. Taking $f \equiv 1$ we see that $X \prec X$ holds i.e., Q.1 holds. Let $A \prec B$ hold. Then there is a τ-l.s.c. and $\mathcal{V}$-u.s.c. function $f : X \to [0, 1]$ such that $f(A) = 1$ and $f(X \setminus B) = 0$; so $A \cap (X \setminus B) = \phi$ and as such $A \subset B$. Hence Q.2 is established. For Q 3 suppose $A \subset B \prec C \subset D$. Then there is a τ-l.s.c and $\mathcal{V}$-u.s.c function $f : X \to [0, 1]$ such that $f(B) = 1$ and $f(X \setminus C) = 0$. But $X \setminus D \subset X \setminus C$ and $A \subset B$. So we have $f(A) = 1$ and $f(X \setminus D) = 0$ showing that $A \prec D$. Next let $A \prec B_k (k = 1, 2, \ldots, n)$ and consequently there are τ-l.s.c. and $\mathcal{V}$-u.s.c. functions $f_1, f_2, \ldots, f_n$ over X satisfying $f_k(A) = 1$ and $f_k(X \setminus B_k) = 0$ for $k = 1, 2, \ldots, n$. Put $f = \min\{f_1, f_2, \ldots, f_n\}$. Then f is a τ-l.s.c. and $\mathcal{V}$-u.s.c. function over X such that $f(A) = 1$ and $f\left(\bigcup_{k=1}^{n} (X \setminus B_k)\right) = f(X \setminus \bigcap_{k=1}^{n} B_k) = 0$ and hence $A \prec \bigcap_{k=1}^{n} B_k$, and Q 4 is established. Q 5 is similarly established. Finally for Q 6 let $A \prec B$; so we have a τ-l.s.c. and $\mathcal{V}$-u.s.c. function f over X such that $f(A) = 1$ and $f(X \setminus B) = 0$. Take $0 < a'' < a' < a < a_1 < a_2 < 1$ and then the set $C = \{x : f(x) > a\}$ is τ-open ; $V_t = \{x : f(x) > a_1\}$ is τ-open and $U_s = \{x : f(x) \geqslant a_2\}$ is $\mathcal{V}$-closed. Also $A \subset U_s \subset V_t \subset C$. This is similar to the main step in the proof of Lemma 2.2, and proceeding as therein we obtain a τ-l.s.c, and $\mathcal{V}$-u.s.c. function $\varphi : X \to [0, 1]$ such that $\varphi(A) = 1$ and $\varphi(X \setminus C) = 0$ and hence $A \prec C$. Also the sets,

$$X \setminus C = \{x : f(x) \leqslant a\} \text{ is } \tau\text{-closed ;}$$

$$U_p = \{x : f(x) \leqslant a''\} \text{ is } \tau\text{-closed;}$$

and $$U_q = \{x : f(x) < a'\} \text{ is } \mathcal{V}\text{-open ;}$$

further $$X \setminus B \subset U_p \subset U_q \subset X \setminus C.$$

So, by the same argument we get a $\mathcal{V}$-u.s.c and τ-l.s.c function $\theta : X \to [0, 1]$ such that $\theta(X \setminus B) = 0$ and $\theta(C) = 1$ and hence $C \prec B$. So $A \prec C \prec B$ holds i.e., Q.6 is established. Thus $\prec$ becomes a quasi-proximity on X.

It remains to show that $\tau_{\prec} = \tau$ and $\tau_{\prec^*} = \mathcal{V}$, $\prec^*$ being conjugate to $\prec$. Let G be a τ-open set. For $x \in G$, using pairwise complete regularity we find a τ-l.s.c. and $\mathcal{V}$-u.s.c. function $f : X \to [0, 1]$ such that $f(x) = 1$ and $f(X \setminus G) = 0$. So $\{x\} \prec G$ i.e., $G \in \mathcal{N}_x$ (w.r.t. $\prec$) for every $x \in G$; Hence $G \in \tau_{\prec}$. Conversely, let $A \in \tau_{\prec}$;

then $\{x\} \lessdot A$ for $x \in A$ and hence there is a τ-l.s.c and $\mathcal{V}$-u.s.c function $g : X \to [0, 1]$ such that $g(x) = 1$ and $g(X \setminus A) = 0$; By τ-l.s.c. property of g we find a τ-neighbourhood N of x such that $g(y) > \frac{1}{2}$ whenever $y \in N$. Clearly then $N \cap (X \setminus A) = \phi$, showing that $N \subset A$; i.e. A is a τ-neighbourhood of x. Hence A is τ-open. Consequently $\tau = \tau_{\lessdot}$. By a similar procedure we arrive at $\mathcal{V} = \tau_{\lessdot^*}$. On the other hand suppose there is a quasi-proximity $\lessdot$ on X such that $\tau = \tau_{\lessdot}$ and $\mathcal{V} = \tau_{\lessdot^*}$. Let O be any member of $\tau_{\lessdot} = \tau$. If $a \in O$, then $\{a\} \lessdot O$, and by Lèmma 2.2 there is a $\tau(= \tau_{\lessdot})$-u.s,c. and $\mathcal{V}(= \tau_{\lessdot^*})$-l.s.c function h: $X \to [0, 1]$ such that $h(a) = 0$ and $h(X \setminus O) = 1$. Hence by Definition 2.1, it follows that τ is completely regular with respect to $\mathcal{V}$. By a similar argument we can show that $\mathcal{V}$ is completely regular with respect to τ. Hence $(X, \tau, \mathcal{V})$ is pairwise completely regular.

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