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ON CONTINUABILITY AND BOUNDEDNESS OF SOLUTIONS OF CERTAIN SECOND ORDER INTEGRO-DIFFERENTIAL EQUATIONS

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The purpose of this paper is to provide some sufficient conditions which ensure the continuability and boundedness of solutions of the integro-differential equation

$$(a(t)\dot{x}) + h(t, x, \dot{x}) + q(t)\, f(x)\, g(\dot{x})$$

$$= e(t, x, \dot{x}) + g_1(\dot{x})\, K\Big(t, \dot{x}, \int_{t_0}^{t} H(t, s, \dot{x})\, ds\Big).$$

Our results include some of the results of Graef and Spikes. The proofs are based on a lemma by Pachpatte. Some illustrative examples are given.

1. INTRODUCTION

In a recent paper Pachpatte (1976) considered the equation

$$(\alpha) \qquad (a(t)\,\dot{x})^* + q(t) f(x)\, g(\dot{x}) = h\Big(t, \dot{x}, \int_{t_0}^{t} K(t, s, \dot{x})\, ds\Big), \quad \Big(* = \frac{d}{dt}\Big)$$

and in the proof of the results established he used Theorem 2. His main aim was to generalise some of the results of Graef and Spikes (1975). However there appears to be an error in the proofs of Theorems 1 and 2 of Pachpatte (1977). The difficulty is that the constant $M_1 > 0$ obtained in the inequality

$$(\alpha_1) \qquad m\, \frac{p(t)}{q(t)} + nV(t_0) \leqslant M_1$$

[see Pachpatte 1977, p. 1065, lines 6 and 10] depends on the solution $(x(t), y(t))$ of system (2) chosen at the beginning of the proof and has no relation to the constant M_1 appearing in the inequality

$$(\alpha_2) \qquad M_1 \int_{t_0}^{t} n\, \frac{c(\tau)}{a(\tau)} \exp\Big(\int_{t_0}^{\tau} w(\rho)\, d\rho \Big)\, d\tau < 1$$

[see Pachpatte 1977, p. 1064, condition (7)]. Therefore, the Lemma given by Pachpatte (1977) cannot be applied. We may note that, using the notation of Pachpatte and the definition of V, we have

$$\begin{aligned} V(t_0) &= V(t_0, x(t_0), y(t_0)) \\ &= p(t_0)\left[\frac{F(x_0) + M}{a(t_0)} + \frac{G(y(t_0))}{q(t_0)}\right] \\ &\geqslant \frac{G(y(t_0))}{q(t_0)} \end{aligned}$$

and thus from the condition $G(y) \to \infty$ as $|y| \to \infty$, we see that choosing the solution $(x(t), y(t))$ with $y(t_0)$ sufficiently large forces the constant M_1 in (α_1) to be arbitrarily large. Hence inequality (α_2) cannot be satisfied for all M_1 obtained from (α_1) unless (α_2) holds for every $M_1 > 0$. But clearly (α_2) cannot hold for every $M_1 > 0$ since the coefficient of M_1 in (α_2) is a positive increasing function of t. We may add that comments similar to above were communicated to us by a referee regarding one of our submissions.

We consider instead of equation (α) the following integro-differential equation:

$$(a(t)\,\dot{x})^{*} + h(t, x, \dot{x}) + q(t) f(x)\, g(\dot{x}) = e(t, x, \dot{x}) + g_1(\dot{x})\, K\left(t, \dot{x}, \int_{t_0}^{t} H(t, s, \dot{x})\, ds\right) \quad \ldots(1)$$

and derive some results which reduce to some of the results of Graef and Spikes (1978) when $g_1 = 0$.

Our main results are given in section 2. In section 3 we give some illustrative examples.

In the sequel it is assumed that

$$a, q : [t_0, \infty) \to (0, \infty)$$

$$f, g, g_1 : R \to \mathrm{R}$$

and $$e, h, K, H : [t_0, \infty) \times R^2 \to R$$

are continuous functions and $g(u) > 0$.

We may note that eqn. (1) is equivalent to the system:

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= \frac{1}{a(t)}\Big[-\dot{a}(t)\, y - h(t, x, y) - q(t) f(x) g(y) + e(t, x, y) \\ &\qquad + g_1(y)\, K\Big(t, y, \int_{t_0}^{t} H(t, s, y)\, ds\Big)\Big]. \end{aligned}\right\} \quad \ldots(2)$$

We let $\dot{q}_+(t) = \max\{\dot{q}(t), 0)\}$ and $\dot{q}_-(t) = \max\{-\dot{q}(t), 0\}$ so that

$$\dot{q}(t) = \dot{q}_+(t) - \dot{q}_-(t).$$

Define

$$F(x) = \int_0^x f(u)\,du$$

$$G(y) = \int_0^y \frac{u\,du}{g(u)}$$

and

$$p(t) = \exp\left(-\int_{t_0}^t \frac{\dot{q}_-(s)}{q(s)}\,ds\right).$$

Assume that there exist nonnegative constants K_1, m and n, and a continuous function $r : [t_0, \infty) \to [0, \infty)$, such that

$$\frac{|y|}{g(y)} \leqslant m + nG(y) \qquad \ldots(3)$$

$$0 \leqslant \frac{ug_1(u)}{g(u)} \leqslant K_1 \qquad \ldots(4)$$

$$|e(t, x, y)| \leqslant r(t) \qquad \ldots(5)$$

$$yh(t, x, y) \geqslant 0. \qquad \ldots(6)$$

Suppose that the functions K and H satisfy

$$|K(t, y, u)| \leqslant c_1(t)\, F_1\left(\frac{|y|}{g(y)}\right) + c_2(t)\,|u| \qquad \ldots(7)$$

$$|H(t, s, y)| \leqslant c_3(t)\,\frac{|y|}{g(y)},\ t, s \in I = [t_0, \infty) \qquad \ldots(8)$$

where $c_i(t)$, $i = 1, 2, 3$ are real-valued, nonnegative continuous functions and $F_1(u)$ is a positive, continuous, monotonic nondecreasing and submultiplicative function for $u > 0$ with $F_1(0) = 0$ and such that

$$\Omega(u) = \int_{u_0}^u \frac{ds}{F_1(s)} \to \infty \text{ as } u \to \infty. \qquad \ldots(9)$$

We will have an occasion to use the following Lemma due to Pachpatte (1975).

Lemma 1.1 — Let $x(t)$, $f(t)$, $g(t)$ and $h(t)$ be real-valued positive continuous functions defined on $I = [0, \infty)$, $W(u)$ be a positive continuous, monotonic, non-decreasing and submultiplicative function for $u > 0$ with $W(0) = 0$, and suppose further that the inequality

$$x(t) \leqslant x_0 + \int_0^t f(s)\, x(s)\, ds + \int_0^t f(s) \left(\int_0^s g(\tau)\, x(\tau)\, d\tau \right) ds$$

$$+ \int_0^t h(s)\, W(x(s))\, ds$$

is satisfied for all $t \in I$, where x_0 is a positive constant. Then

$$x(t) \leqslant \Omega_1^{-1} \left[\Omega_1(x_0) + \int_0^t h(s)\, W(b(s))\, ds \right] \cdot b(t), \quad 0 \leqslant t \leqslant c$$

where

$$b(t) = 1 + \int_0^t f(s) \exp \left(\int_0^s [f(\tau) + g(\tau)]\, d\tau \right) ds$$

$$\Omega_1(r) = \int_{r_0}^r \frac{ds}{W(s)}, \; r \geqslant r_0 > 0$$

and Ω_1^{-1} is the inverse function of Ω_1 and $t \in [0, c] \subset I$, so that

$$\Omega_1(x_0) + \int_0^t h(s)\, W(b(s))\, ds \in \mathrm{Dom}\, (\Omega_1^{-1}).$$

Remark: The result of the lemma holds for all intervals of the form (t_0, ∞), $t_0 \geqslant 0$.

2. Main Results

Theorem 2.1 — If conditions (3) – (9) hold,

$\dot{a}(t) \geqslant 0,$

$F(x)$ is bounded from below

and

$$G(y) \to \infty \text{ as } |y| \to \infty \qquad \text{...(10)}$$

then all solutions of (2) are defined for all $t \geqslant t_0$.

PROOF : Suppose that $(x(t), y(t))$ is a solution of (2) and $T > t_0$ such that

$$\lim_{t \to T^-} [\,|x(t)| + |y(t)|\,] = \infty.$$

Since $F(x)$ is bounded from below, $F(x) \geqslant -K_2$ for some $K_2 > 0$.
Consider a function V defined by

$$V(t, x, y) = p(t)\left[\frac{F(x) + K_2}{a(t)} + \frac{G(y)}{q(t)}\right]$$

then

$$\dot{V}(t) = p(t)\Big[-\frac{\dot{q}_-(t)}{q(t)}\frac{F(x)+K_2}{a(t)} - \frac{\dot{q}_-(t)}{q(t)}\frac{G(y)}{q(t)} - \frac{\dot{a}(t)}{a(t)}\frac{F(x)+K_2}{a(t)}$$

$$+\frac{f(x)\,y}{a(t)} - \frac{\dot{q}(t)}{q(t)}\frac{G(y)}{q(t)} + \frac{1}{a(t)\,q(t)}\frac{y}{g(y)}\Big(-\dot{a}(t)\,y$$

$$- h(t, x, y) - q(t)\,f(x)\,g(y) + e(t, x, y)$$

$$+ g_1(y)\,K\Big(t, y, \int_{t_0}^{t} H(t, s, y)\,ds\Big)\Big)\Big]$$

$$\leqslant p(t)\Big[\frac{r(t)}{a(t)\,q(t)}\frac{|y|}{g(y)} + \frac{K_1}{a(t)\,q(t)}\Big(c_1(t)\,F_1\Big(\frac{|y|}{g(y)}\Big)$$

$$+ c_2(t)\int_{t_0}^{t} c_3(s)\frac{|y|}{g(y)}\,ds\Big].$$

Let us put $u(t) = \dfrac{p(t)}{q(t)}\dfrac{|y|}{g(y)}$, then

$$u(t) \leqslant \frac{p(t)}{q(t)}\,(m + n\,G(y))$$

$$\leqslant \frac{mp(t)}{q(t)} + nV(t)$$

$$= \frac{mp(t)}{q(t)} + nV(t_0) + n\int_{t_0}^{t} \dot{V}(s)\,ds.$$

Since $\dfrac{p(t)}{q(t)}$ is bounded on $[t_0, T]$, there exists a constant $M > 0$ such that

$$\frac{mp(t)}{q(t)} + nV(t_0) \leqslant M.$$

Using the inequality involving $\dot{V}(t)$ we obtain the following estimate for $u(t)$:

$$u(t) \leqslant M + \int_{t_0}^{t} \frac{nr(s)}{a(s)} u(s)\, ds$$

$$+ \int_{t_0}^{t} nK_1 \frac{c_1(s)}{a(s)} \frac{p(s)}{q(s)} F_1\left(\frac{q(s)}{p(s)} u(s)\right) ds$$

$$+ \int_{t_0}^{t} nK_1 \frac{c_2(s)\, p(s)}{a(s)\, q(s)} \left(\int_{t_0}^{s} \frac{c_3(\tau)\, q(\tau)}{p(\tau)} u(\tau)\, d\tau \right) ds.$$

We define $\beta_i(t)$, $i = 1, 2, 3$ by

$$\beta_1(t) = \max \left\{ \frac{nr(t)}{a(t)}, nK_1 \frac{c_2(t)\, p(t)}{a(t)\, q(t)} \right\}$$

$$\beta_2(t) = nK_1 \frac{c_1(t)}{a(s)} \frac{p(t)}{q(t)} F_1\left(\frac{q(t)}{p(t)}\right)$$

and

$$\beta_3(t) = \frac{c_3(t)\, q(t)}{p(t)}$$

and write

$$u(t) \leqslant M + \int_{t_0}^{t} \beta_1(s)\, u(s) + \int_{t_0}^{t} \beta_1(s) \left(\int_{t_0}^{s} \beta_3(\tau)\, u(\tau)\, d\tau \right) ds$$

$$+ \int_{t_0}^{t} \beta_2(s)\, F_1(u(s))\, ds.$$

Now an application of Lemma 1.1 yields

$$u(t) \leqslant \Omega^{-1} \left[\Omega(M) + \int_{t_0}^{t} \beta_2(s)\, F_1(A(s))\, ds \right]. \; A(t) \leqslant K_3 < \infty$$

for $t \in [t_0, T)$, for some constant K_3, where

$$A(t) = 1 + \int_{t_0}^{t} \beta_1(s) \exp\left(\int_{t_0}^{s} [\beta_1(\tau) + \beta_3(\tau)]\, d\tau \right) ds.$$

Hence

$$\frac{p(t)}{q(t)}\, G(y) \leqslant V(t) \leqslant V(t_0) + \int_{t_0}^{t} K_3\beta_1(s)\, ds$$

$$+ \int_{t_0}^{t} \beta_1(s)\left(\int_{t_0}^{s} K_3\beta_3(\tau)\, d\tau\right) ds + \int_{t_0}^{t} F_1(K_3)\, \beta_2(s)\, ds$$

$$\leqslant K_4 < \infty \qquad \text{on } [t_0, T), \quad \text{for some} \quad K_4 > 0.$$

Thus $G(y(t))$ is bounded on $[t_0, T)$ and hence

$$y(t) = \dot{x}(t)$$

is bounded on $[t_0, T)$. An integration shows that $x(t)$ is bounded $[t_0, T)$ and so we have a contradiction to the assumption that $(x(t), y(t))$ is a solution of (2) with finite escape time.

Remark : If $g_1 = 0$, Theorem 1 of Graef and Spikes (1978) is included in Theorem 2.1.

Theorem 2.2 — Suppose (3) – (9) hold, $\dot{a}(t) \geqslant 0$ and $a(t)$ is bounded,

$$F(x) \to \infty \quad \text{as} \quad |x| \to \infty \qquad \ldots(11)$$

and

$$\frac{\dot{q}_-(t)}{q(t)},\ \beta_i(t), \quad i = 1, 2, 3 \quad \text{are in} \quad \mathcal{L}(t_0, \infty) \qquad \ldots(12)$$

(with $\beta_i(t)$ as in the previous Theorem).

Then all solutions of (1) are bounded. In addition if $q(t)$ is bounded from above and if (10) holds, then all solutions of (2) are bounded.

PROOF : Since $F(x) \to \infty$ as $|x| \to \infty$, $F(x)$ is bounded from below, say $F(x) \geqslant -K_2$ for some $K_2 > 0$. Letting $V(t)$ be as in the proof of Theorem 2.1, differentiating and integrating and using conditions (3) – (9) we obtain

$$u(t) \leqslant \frac{mp(t)}{q(t)} + nV(t_0) + \int_{t_0}^{t} \beta_1(s)\, u(s) + \int_{t_0}^{t} \beta_1(s)\left(\int_{t_0}^{s} \beta_3(\tau)\, u(\tau)\, d\tau\right) ds$$

$$+ \int_{t_0}^{t} \beta_2(s)\, F_1(u(s))\, ds$$

where $u(t) = \frac{p(t)}{q(t)} \frac{|y|}{g(y)}$.

Now $p(t) \leqslant 1$ and (12) bounds $q(t)$ away from zero, so

$$\frac{mp(t)}{q(t)} + nV(t_0) \leqslant M_1 \qquad \text{for some} \quad M_1 > 0.$$

Then using Lemma, we have

$$u(t) \leqslant \Omega^{-1}\left[\Omega(M_1) + \int_{t_0}^{t} \beta_2(s)\, F_1(A(s))\, ds\right] . A(t), \quad t \in [t_0, T)$$

where

$$A(t) = 1 + \int_{t_0}^{t} \beta_1(s) \exp\left(\int_{t_0}^{s} [\beta_1(\tau) + \beta_3(\tau)]\, d\tau\right) ds.$$

Using (12), we have

$$u(t) = \frac{p(t)\,|\,y\,|}{q(t)\,g(y)} \leqslant K_3 < \infty \qquad \text{for some} \quad K_3 > 0$$

and thus

$$V(t) \leqslant M_1 + \int_{t_0}^{t} K_3\beta_1(s)\, ds + \int_{t_0}^{t} \beta_1(s)\left(\int_{t_0}^{s} K_3\beta_3(\tau)\, d\tau\right) ds$$
$$+ \int_{t_0}^{t} F_1(K_3)\, \beta_2(s)\, ds \leqslant K_4 < \infty$$

and hence

$$F(x) \leqslant K_4 a(t) \leqslant K_5 < \infty$$

for some constants K_4 and K_5. Thus $x(t)$ is bounded. Moreover

$$G(y) \leqslant K_4 q(t) \leqslant K_6 < \infty$$

for some constant K_6, consequently the conclusion follows.

Remark : We can drop the condition on $\dot{a}(t)$ by requiring a stronger condition on $g(y)$, namely, that there are positive constants M_1 and l such that

$$\frac{y^2}{g(y)} \leqslant M_1 G(y) \quad \text{for} \quad |\,y\,| \geqslant l. \qquad \ldots(13)$$

The proof of this result involves more details than given in the proof of Theorem 2.1, so we omit it noting only that (13) implies (3).

Theorem 2.3 — Suppose (4) - (9), (11), (13) hold,

$$0 < q_1 \leqslant q(t) \leqslant q_2, \quad \ldots(14)$$

$$\beta(t), \quad C_3(t) \quad \text{and} \quad \frac{C_1(t)}{a(t)} \quad \text{are in} \quad \mathcal{L}(t_0, \infty) \quad \ldots(15)$$

where

$$\beta(t) = \max \left\{ M_1 K_1 \frac{C_2(t)}{a(t)}, \ \frac{\dot{q}(t)}{q(t)} + M_1 \frac{r(t)}{a(t)} + (M_1 + 1) \frac{\dot{a}_-(t)}{a(t)} \right\}$$

and $a(t) \leqslant a_1$,

then all solutions of (1) are bounded. In addition if (10) holds, then all solutions of (2) are bounded.

PROOF: Condition (13) implies that there exists $A > 0$ such that

$$\frac{y^2}{g(y)} \leqslant A + M_1 G(y) \quad \text{for all } y.$$

Notice also that if $|y| \leqslant 1$, then

$$\frac{|y|}{g(y)} \leqslant B \qquad \text{for some} \quad B > 0$$

and if $|y| > 1$, then

$$\frac{|y|}{g(y)} \leqslant \frac{y^2}{g(y)},$$

so
$$\frac{|y|}{g(y)} \leqslant B + \frac{y^2}{g(y)} \qquad \text{for all } y.$$

By (11), $F(x) \geqslant -K_2$ for some $K_2 > 0$.

If $M_1 \geqslant 1$, define

$$V(t, x, y) = \frac{q(t)}{a(t)} (F(x) + K_2) + G(y) + A + B,$$

then

$$\dot{V}(t) = \frac{\dot{q}(t)}{a(t)} (F(x) + K_2) - \frac{\dot{a}(t)}{a(t)} \frac{q(t)}{a(t)} (F(x) + K_2)$$
$$+ \frac{q(t)}{a(t)} f(x)\, y + \frac{y}{a(t)\, g(y)} \Big[-\dot{a}(t)\, y - h(t, x, y) - q(t) f(x)\, g(y) +$$

(*equations continued on p.* 959)

$$+ e(t, x, y) + g_1(y) K\left(t, y, \int_{t_0}^{t} H(t, s, y)\, ds\right)\Bigg].$$

$$\leqslant \frac{\dot{g}(t)}{g(t)}\left[\frac{q(t)}{a(t)}(F(x) + K_2)\right]$$

$$+ \frac{\dot{a}_-(t)}{a(t)}\left[\frac{q(t)}{a(t)}(F(x) + K_2) + A + M_1G(y)\right]$$

$$+ \frac{r(t)}{a(t)}(A + B + M_1G(y))$$

$$+ \frac{K_1}{a(t)}\left[C_2(t)\int_{t_0}^{t} C_3(s)\,[A + B + M_1G(y)]\, ds\right.$$

$$\left. + C_1(t) F_1(A + B + M_1G(y))\right].$$

$$\leqslant \left(\frac{\dot{q}(t)}{q(t)} + M_1\frac{r(t)}{a(t)} + (M_1 + 1)\frac{\dot{a}_-(t)}{a(t)}\right) V(t)$$

$$+ K_1\frac{C_2(t)}{a(t)}\int_{t_0}^{t} M_1C_3(s)\, V(s)\, ds$$

$$+ \frac{K_1C_1(t)}{a(t)} F_1(M_1)\, F_1(V(t)).$$

Integrating from t_0 to t we have

$$V(t) \leqslant V(t_0) + \int_{t_0}^{t} \beta(s)\, V(s)\, ds + \int_{t_0}^{t} \beta(s)\left(\int_{t_0}^{s} C_3(\tau)\, V(\tau)\, d\tau\right) ds$$

$$+ \int_{t_0}^{t} K_1F_1(M_1)\frac{C_1(s)}{a(s)} F_1(V(s))\, ds.$$

Applying Lemmas we have,

$$V(t) \leqslant \Omega^{-1}\left[\Omega(V(t_0)) + \int_{t_0}^{t} K_1F_1(M_1)\frac{C_1(s)}{a(s)} F_1\left\{1 + \right.\right.$$

(*equation continued on p.* 960)

$$+ \int_{t_0}^{t} \beta(\tau) + \exp\left(\int_{t_0}^{\tau} [\beta(\xi) + C_3(\xi)]\, d\xi \right) d\tau \Big\} \, ds \Big]$$

$$\times \left[1 + \int_{t_0}^{t} \beta(s) \exp\left\{ \int_{t_0}^{s} (\beta(\tau) + C_3(\tau))\, d\tau \right\} ds \right].$$

By (15), $V(t)$ is bounded, say $V(t) \leqslant K_3 < \infty$, for some constant K_3.

Hence

$$F(x(t)) \leqslant K_3 \frac{a(t)}{q(t)} \leqslant \frac{K_3 a_1}{q_1} = K_4$$

so $F(x)$ is bounded for $t \geqslant t_0$. The boundedness of $x(t)$ follows from (11). Moreover

$$G(y) \leqslant K_5 \quad \text{for some constant } K_5,$$

Consequently the conclusion follows from (10).

If $M_1 < 1$, define

$$V(t, x, y) = \frac{q(t)}{a(t)} (F(x) + K_2) + G(y) + \frac{A + B}{M_1}$$

then

$$\begin{aligned} \dot{V}(t) \leqslant & \frac{\dot{q}(t)}{q(t)} \left[\frac{q(t)}{a(t)} (F(x) + K_2) \right] \\ & + \frac{\dot{a}_-(t)}{a(t)} \left[\frac{q(t)}{a(t)} (F(x) + K_2) + A + M_1 G(y) \right] \\ & + \frac{r(t)}{a(t)} [A + B + M_1 G(y)] \\ & + \frac{K_1}{a(t)} \left[C_2(t) \int_{t_0}^{t} C_3(s) \frac{|y|}{g(y)}\, ds + C_1(t) F_1\left(\frac{|y|}{g(y)} \right) \right] \\ \leqslant & \left(\frac{\dot{q}(t)}{q(t)} + M_1 \frac{r(t)}{a(t)} + (M_1 + 1) \frac{\dot{a}_-(t)}{a(t)} \right) V(t) \\ & + \frac{K_1 M_1 C_2(t)}{a(t)} \int_{t_0}^{t} C_3(s)\, V(s)\, ds + \frac{K_1}{a(t)} F_1(M_1)\, C_1(t)\, F_1(V(t)). \end{aligned}$$

The remainder of the proof is similar to the one for the case $M_1 \geqslant 1$.

Remark : If $g_1 = 0$, $r(t) = \dfrac{a(t)\,\dot{q}(t)}{M_1 q(t)}$, $a(t)$ and $q(t)$ are bounded from above and $\int\limits_{t_0}^{\infty} \dfrac{\dot{a}_-(s)}{a(s)}\,ds < \infty$, then conditions (4), (7), (8), (9), (14), (15) can be discarded. In that case Theorem 2.3 becomes Theorem 3 and Corollary 4 of Graef and Spikes (1978).

3. Illustrative Examples

1. Consider the equation

$$(a(t)\,\dot{x})^* + \frac{\dot{x}}{1+\dot{x}^2}\,e^{-tx} + q(t)\,f(x)$$

$$= r(t)\sin t\dot{x} + \frac{x}{1+\dot{x}^2}\left[C_1(t)\,(\dot{x})^\alpha + C_2(t)\int\limits_0^t C_3(s)\,\dot{x}\tanh s\dot{x}\,ds\right]$$

where $\alpha \in (0, 1]$, a, q, r, C_1, C_2 and C_3 satisfy the hypotheses of our Theorems. Let $f(x)$ be one of the following functions

$$x^{2n+1},\quad n = 0, 1, \ldots, m \quad \text{or} \quad e^x \quad \text{or} \quad \sinh x,$$

then $F(x)$ is $\dfrac{x^{2n+2}}{2n+2}$, $n = 0, 1, \ldots, m$ or e^x or $\cosh x$ respectively and is bounded from below and $F(x) \to \infty$ as $|x| \to \infty$,

$$h(t, x, \dot{x}) = \frac{\dot{x}}{1+\dot{x}^2}\,e^{-tx} \quad \text{so} \quad yh(t, x, y) \geqslant 0.$$

$$g(\dot{x}) = 1, \qquad G(y) = \frac{y^2}{2}$$

so $$\frac{|y|}{g(y)} = |y| \leqslant \tfrac{1}{2} + \tfrac{1}{2}y^2 \quad \text{i.e.} \quad m = \tfrac{1}{2},\quad n = 1$$

or $$\frac{y^2}{g(y)} = y^2 \leqslant 0 + 2(\tfrac{1}{2}y^2) \quad \text{i.e.} \quad A = 0,\quad M_1 = 2$$

$$g_1(\dot{x}) = \frac{\dot{x}}{1+\dot{x}^2} \quad \text{so} \quad \frac{\dot{x}g_1(\dot{x})}{g(\dot{x})} = \frac{\dot{x}^2}{1+\dot{x}^2} \leqslant 1 = K_1$$

$$|K(\ldots)| \leqslant C_1(t)\,|\dot{x}^\alpha| + C_2(t)\int\limits_0^t C_3(s)\,|\dot{x}|\,ds$$

$$= C_1(t)\left(\frac{|y|}{g(y)}\right)^\alpha + C_2(t)\int\limits_0^t C_3(s)\,\frac{|y|}{g(y)}\,ds.$$

Thus we conclude that all solutions of the above equation can be defined for $t \geqslant 0$ and are bounded. We believe that such a conclusion for this equation is not deducible from any other known criteria.

2. Consider the equation

$$(a(t)\,\dot{x})^{\cdot} + \frac{\dot{x}^3}{1+\dot{x}^2}\sin^2 tx + q(t)\,[\tanh \dot{x}^2 + \operatorname{sech} \dot{x}^2]\,f(x)$$

$$= r(t)\sin(tx\dot{x}) + g_1(\dot{x})\left[C_1(t)\,(\dot{x})^{\alpha} + C_2(t)\int_0^t C_3(s)\,\dot{x}ds\right]$$

where α, a, q, r, g_1, C_1, C_2 and C_3 can be chosen to satisfy the hypotheses of our Theorems. Let $f(x)$ be as in the previous example. We let

$$h(t, x, \dot{x}) = \frac{\dot{x}^3}{1+\dot{x}^2}\sin^2 tx.$$

$$g(y) = \tanh y^2 + \operatorname{sech} y^2, \quad G(y) = \tfrac{1}{2}\log(1+\sinh y^2).$$

We note that

$$\frac{|y|}{g(y)} = \frac{|y|\cosh y^2}{1+\sinh y^2} = \frac{|y|\,[e^{2y^2}+1]}{e^{2y^2}+2e^{y^2}-1}$$

$$\geqslant \frac{|y|\,(e^{2y^2}+1)}{3(e^{2y^2}+1)} = \frac{|y|}{3}.$$

Now it is possible to find m and n nonnegative constants such that

$$\frac{|y|}{g(y)} \leqslant m + \frac{n}{2}\log(1+\sinh y^2)$$

or

$$\frac{y^2}{g(y)} \leqslant m_1 + \frac{n_1}{2}\log(1+\sinh y^2),$$

m_1 and n_1 are nonnegative constants.

Also

$$|\,e(t, x, \dot{x})\,| = |\,r(t)\sin tx\dot{x}\,| \leqslant r(t)$$

and

$$|\,K(\ldots)\,| \leqslant C_1(t)\,(\,|\,\dot{x}\,|\,)^{\alpha} + C_2(t)\int_0^t C_3(s)\,\dot{x}\,ds$$

$$\leqslant 3^{\alpha}C_1(t)\left(\frac{|y|}{g(y)}\right)^{\alpha} + C_2(t)\int_0^t 3C_3(s)\left(\frac{|y|}{g(y)}\right)ds.$$

Thus all the hypotheses of our Theorems are satisfied and hence we conclude that all solutions of the above equation can be defined for $t \geqslant 0$ and are bounded. We add that such a conclusion is not deducible from any other known criteria.

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