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THE SPECTRUM OF A CLASS OF INTEGRAL TRANSFORMS

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Let I_μ, μ some real number, denote the class of linear transformations S which are continuous on L^p into $L^q(1 < p, q < \infty)$ and which are such that $\int_0^u Sf(t)\,dt = \int_{-\infty}^{\infty} |x|^{\mu-2} (\operatorname{sgn} x)\, k(ux)\, f(x)\, dx$. In general, i.e. unless $\mu = 1$ and $p = q = 2$, the spectrum of $S \in I_\mu$ is not defined. Here a study of the sets $\{\alpha \in \mathbb{C} : Sf = \alpha f$ has a unique non-trivial solution $f \in L^p \cap L^q\}$ and $\{\alpha \in \mathbb{C} : Sf - \alpha f = g$, for each non-trivial $g \in L^p \cap L^q$, has a unique non-trivial solution $f \in L^p \cap L^q\}$ is carried out by using the 'permutability' properties of mappings $S \in I_\mu$ and a related class of operators. The particular case, in which the spectrum is defined, is deduced from the main result.

1. INTRODUCTION

Let $L^p = L^p(-\infty, \infty)$, $L_p = L^p(0, \infty)$, and let $C(L^p, L^q)$ denote the class of linear transformations which are continuous on L^p into L^q. We say that the linear transformation $S \in C(L^p, L^q)$, $1 < p, q < \infty$, belongs to the class I_μ, μ some real number, if there exists a Lebesgue measurable function k on $(-\infty, \infty)$ such that

$$\int_0^u Sf(t)\, dt = \int_{-\infty}^{\infty} |x|^{\mu-2} (\operatorname{sgn} x)\, k(ux)\, f(x)\, dx \qquad \ldots(1)$$

where $-\infty < u < \infty$. Mappings $S \in I_\mu$ satisfy the functional equation

$$St(a) = |a|^{-\mu}\, t(a^{-1})\, S, \quad -\infty < a < \infty, \quad a \neq 0, \qquad \ldots(1a)$$

where $t(a)$ is the operator $(t(a) f)(x) = f(ax)$. Conversely, if $S \in I_\mu \cap C(L^p, L^q)$ satisfies functional equation ($1a$), then it can be shown under fairly mild hypotheses on μ (see Duggal 1978a, Lemma 1) that S has the integral representation (1).

A number of the important integral transforms encountered in applications belong to the class I_μ. In particular, the Fourier transform, obtained from (1) upon setting $\mu = 1$ and $k(ux) = i(1 - e^{iux})$ [or alternatively, upon assuming that differentiation under the integral sign is a continuous operation, and setting $\mu = 1$ and $(d/du)\, k(ux) = xe^{iux}$], and the so called weighted Fourier transforms (see Okikiolu 1971) belong to I_μ. In the case in which $\mu = 1$, the class of mappings $S \in I_\mu$ which are continuous on L_2 into itself consists of Watson transforms (see de Snoo 1973). Let $S \in I_\mu \cap C(L^p, L^q)$.

Let $\sigma_P(S)$ denote the set of complex numbers α such that $Sf = \alpha f$ has a non-trivial solution $f \in L^p \cap L^q$, and let $\rho(S)$ denote the set of complex numbers α such that $Sf - \alpha f = g$ has, for each non-trivial $g \in L^p \cap L^q$, a unique non-trivial solution $f \in L^p \cap L^q$.

Let S be a Watson transform. Using the theory of Mellin transforms of functions belonging to L_2, a complete characterisation of the sets $\sigma_P(S)$ and $\rho(S)$—which now correspond, respectively, to the point spectrum and the resolvent set of S—has been carried by de Snoo (1974). Our purpose in this note is to study the sets $\sigma_P(S)$ and $\rho(S)$ for a general $S \in I_\mu \cap C(L^p, L^q)$. We show that for $1 < p \leqslant q \leqslant p'$, the complex number $\alpha \in \rho(S)$ (or $\alpha \in \sigma_P(S)$) if and only if $-\alpha \in \rho(S')$ (resp., $-\alpha \in \sigma_P(S')$), where S' denotes the mapping adjoint to S. In the case in which $\mu = 1$, this implies that the sets $\rho(S)$ and $\sigma_P(S)$ are symmetric about the origin. We also give a necessary and sufficient condition for α to belong to $\rho(S)$ or $\sigma_P(S)$, $S \in I_1 \cap C(L^p, L^q)$, and show that this condition leads us to the characterisation in de Snoo (1974) in the case in which S is a Watson transform.

The absence of a complete theory of Mellin transforms, especially the uniqueness theorem, for general values of p and q implies that the technique employed by de Snoo (1974) in the study of the spectrum of Watson transforms cannot be extended to carry out a study of the sets $\rho(S)$ and $\sigma_P(S)$. A different approach is required. In the following we draw from the 'permutability' properties (see Duggal 1978a) of the mappings belonging to the classes I_μ and G_λ (defined in section 2).

2. Further Notation

Throughout the following $1 < p, q < \infty$. Given p, the index conjugate to p will be denoted by p'. The linear mapping R_t, t some real number, will be defined by

$$R_t f(x) = |x|^{-t} (\operatorname{sgn} x) f(x^{-1}).$$

It is clear that $R_t^{-1} = R_t$. We denote R_1 by R. The linear mapping Q_t, t some real number, will be defined by

$$Q_t f(x) = |x|^{-t} f(x).$$

We say that the mapping $T \in C(L^p, L^q)$ belongs to the class G_λ, λ some real number, if there exists a Lebesgue measurable function K on $(-\infty, \infty)$ such that

$$\int_0^u Tf(t)\, dt = \int_{-\infty}^{\infty} |x|^\lambda K(ux^{-1}) f(x)\, dx$$

where $-\infty < u < \infty$. Letting $\lambda = 1 - \mu$, it is easily seen that if $T \in G_\lambda \cap C(L^p, L^q)$, then $S = TR \in I_\mu \cap C(RL^p, L^q)$. Here $R_t L^p$ denotes the space of equivalence classes of Lebesgue measurable functions f on $(-\infty, \infty)$ for which

$$\| f \|^p_{R_t L^p} = \int_{-\infty}^{\infty} | f(x) |^p | x |^{tp-2} dx < \infty.$$

The mappings belonging to the class G_λ satisfy the functional equation

$$Tt(a) = | a |^{-\lambda} (\text{sgn } a)\, t(a)\, T, \qquad -\infty < a < \infty, a \neq 0. \qquad ...(2)$$

Given $T \in G_\lambda \cap C(L^p, L^q)$ (or $S \in I_\mu \cap C(L^p, L^q)$) we say that T (or S) is 0-adjoint if $T' = R_{1-\lambda}\, TR_{1+\lambda} = dT$(resp., $S' = Q_{1-\mu}\, SQ_{\mu-1} = dS$) for some complex constant d such that $| d | = 1$ (see Duggal 1978a, Definition on p. 97). Clearly, each $S \in I_1 \cap C(L^p, L^q)$ is 0-adjoint, and then $S' = S$. Also, if the mapping $T \in G_\lambda \cap C(L^p, L^q)$ is 0-adjoint, then it is continuous on

$$L^{q'} \to L^{p'},\ R_{1+\lambda} L^p \to R_{1-\lambda} L^q, \text{ and } R_{1+\lambda} L^{q'} \to R_{1-\lambda} L^{p'}$$

simultaneously.

3. Results

The following lemma is basic to the technique that we employ in the study of the sets $\rho(S)$ and $\sigma_P(S)$.

Lemma 3.1 — (i) Let $S \in I_\mu \cap C(L^p, L^q)$ and $T \in G_\lambda \cap C(L^q, L^r)$, $1 < p, q, r < \infty$. If $\mu - \lambda = 1$, then

$$TS = S'T' \qquad ...(3)$$

on L^p.

(ii) Let $T \in G_\lambda \cap C(L^p, L^q)$ and $S \in I_\mu \cap C(L^q, L^r)$, $1 < p, q, r < \infty$. If $\lambda + \mu = 1$, then

$$ST = T'S' \qquad ...(4)$$

on L^p. Furthermore, if T is 0-adjoint, then

$$ST = dTS',\ d \text{ as in the definition of 0-adjointness, on } L^{q'}. \qquad ...(4a)$$

(iii) Let $S \in I_1 \cap C(L^p, L^q)$ and $T \in G_0 \cap C(L^p, L^p)$. Then $TRS = RST$ on L^p.

Remark : The hypotheses on μ and λ in (i) and (ii) of the lemma are necessary too (see Duggal 1979, Theorem 2.1). A proof of the first part of (ii) appears in Duggal (1978a), Theorem 3.1, under some additional hypotheses. We note that these additional hypotheses are superfluous in so far as one is interested in proving (4) only. A proof of (4a) follows from that of (4) because of the 0-adjointness of T. (iii) is a particular case of (ii) (see Duggal 1978a, Corollary 6). Although a proof of (i) follows from an argument analogous to that used in the proof of (ii), we produce it here for completeness.

PROOF OF (i) : Since $S \in I_\mu$ and $T \in G_\lambda$, S satisfies functional equation (1*a*) and T satisfies functional equation (2). This, in view of the hypothesis that $\mu - \lambda = 1$, implies that TS satisfies functional equation (1)$'$ with $\mu = 1$. Clearly, $TS \in C(L^p, L^r)$; hence $TS \in I_1 \cap C(L^p, L^r)$. This implies that TS is 0-adjoint, and hence that

$$TS = (TS)' = S'T'.$$

The proof will now be complete once we are able to show that $S'T' \in C(L^p, L^r)$.

Since $S \in I_\mu \cap C(L^p, L^q)$ and $T \in G_\lambda \cap C(L^q, L^r)$, we have that $\mu = p^{-1} + q^{-1}$, $\lambda = q^{-1} - r^{-1}$ and $p \leqslant q \leqslant r$ (see Duggal 1978a, p. 95). This, since $\mu - \lambda = 1$, implies that $p^{-1} + r^{-1} = 1$, and so that $p \leqslant q \leqslant r = p'$. Again, since $T \in C(L^q, L^r)$ and $S \in C(L^p, L^q)$, $T' \in C(L^{r'}, L^{q'})$ and $S' \in C(L^{q'}, L^{p'})$. Hence, because $r = p'$, $S'T' \in C(L^p, L^r)$.

Theorem 3.2 — Let $S \in I_\mu \cap C(L^p, L^q)$, $q \leqslant p'$. The complex number $\alpha \in \rho(S)$ (or $\alpha \in \sigma_P(S)$) if and only if $-\alpha \in \rho(S')$ (resp., $-\alpha \in \sigma_P(S')$).

PROOF : We prove the theorem for $\alpha \in \rho(S)$: the proof for $\alpha \in \sigma_P(S)$ is similar.

The continuity hypotheses on S, and the fact that $q \leqslant p'$, imply that

$$\mu = p^{-1} + q^{-1} \geqslant 1.$$

Let λ be defined by $\lambda = \mu - 1$. Let $T \in G_\lambda \cap C(L^q, L^r)$ be an 0-adjoint mapping such that $T' = R_{1-\lambda}\, TR_{1+\lambda} = -T$. [Such mappings exist : *cf.* Duggal 1978a, (5.1) (ii)]. Then $r = p'$ and TS is well defined as a continuous mapping on L^p into $L^{p'}$.

Suppose that $\alpha \in \rho(S)$. Then to each $g \in L^p \cap L^q$ there exists just one $f \in L^p \cap L^q$ such that

$$Sf - \alpha f = g.$$

Applying T to both sides of this equality it follows that if $\alpha \in \rho(S)$, then to each $g \in L^p \cap L^q$ there exists just one $f \in L^p \cap L^q$ such that

$$TSF - \alpha Tf = Tg.$$

By Lemma 3.1 (i), $TS = S'T' = -S'T$ on L^p. Since $T \in C(L^q, L^{p'})$ is 0-adjoint, $T \in C(L^p, L^{q'})$ simultaneously. Hence, for $f \in L^p \cap L^q$, $Tf \in L^{p'} \cap L^{q'}$. It thus follows that if $\alpha \in \rho(S)$ then to each $h = Tg \in L^{p'} \cap L^{q'}$ there exists just one $F = Tf \in L^{p'} \cap L^{q'}$ such that

$$-S'F - \alpha f = h,$$

or setting $-h = G$, if $\alpha \in \rho(S)$ then to each $G \in L^{p'} \cap L^{q'}$ there exists just one $F \in L^{p'} \cap L^{q'}$ such that

$$S'F + \alpha F = G.$$

Hence if $\alpha \in \rho(S)$, then $-\alpha \in \rho(S')$.

To see the converse, suppose that $\alpha \in \rho(S')$. Then to each $g \in L^{p'} \cap L^{q'}$ there exists just one $f \in L^{p'} \cap L^{q'}$ such that

$$S'f - \alpha f = g.$$

Let $\lambda = \mu - 1$, and let T be the mapping defined above. Since S' satisfies functional equation (1)' with μ replaced by $2 - \mu$ (see Duggal 1978a, p. 96), $T'S' = -TS'$ satisfies functional equation (1a) with μ replaced by $\lambda + (2 - \mu) = 1$. It follows from Lemma 3.1 (ii) that $T'S' = ST$ on $L^{q'}$. Also, for $f \in L^{p'} \cap L^{q'}$, $Tf \in L^p \cap L^q$. Hence, by an argument similar to that before, if $\alpha \in \rho(S')$ then $-\alpha \in \rho(S)$.

The case $\mu = 1$ — It is immediate from the preceding theorem that if

$$S \in I_1 \cap C(L^p, L^q),$$

then the sets $\rho(S)$ and $\sigma_P(S)$ are symmetric about the origin. In fact more can be said. We have:

Theorem 3.3 — Let $S \in I_1 \cap C(L^p, L^q)$. Then

(i) $\alpha \in \rho(S)$ if and only if to each $g \in L^p \cap L^{p'}$ there exists just one $f \in L^p \cap L^{p'}$ such that

$$STf - \alpha T'f = T'g$$

for all $T \in G_0 \cap C(L^p, L^p)$;

(ii) $\alpha \in \sigma_P(S)$ if and only if there exists a non-trivial $f \in L^p \cap L^{p'}$ such that

$$STf = \alpha T'f$$

for all $T \in G_0 \cap C(L^p, L^p)$.

Proof: The proof for both (i) and (ii) is similar: we prove (i). It is clear from the continuity hypotheses on S that $q = p'$.

Suppose $\alpha \in \rho(S)$. Let $T \in G_0 \cap C(L^p, L^p)$ be some given mapping. Then to each $g \in L^p \cap L^{p'}$ there exists just one $f \in L^p \cap L^{p'}$ such that

$$Sf - \alpha f = g.$$

Applying TR to both sides of the equality, and using Lemma 3.1 (iii), we have that if $\alpha \in \rho(S)$ then to each $g \in L^p \cap L^{p'}$ there exists just one $f \in L^p \cap L^{p'}$ such that

$$RSTf - \alpha TRf = TRg$$

or

$$RSTf - \alpha RRTRf = RRTRg.$$

Since $T' = RTR$ (see Duggal 1978a, Lemma 1), it follows that if $\alpha \in \rho(S)$ then to each $g \in L^p \cap L^{p'}$ there exists just one $f \in L^p \cap L^{p'}$ such that

$$RSTf - \alpha RT'f = RT'g,$$

or

$$STf - \alpha T'f = T'g.$$

Suppose now that to each $g \in L^p \cap L^{p'}$ there exists just one $f \in L^p \cap L^{p'}$ such that

$$STf - \alpha T'f = T'g.$$

Since, by Lemma 3.1 (ii), $ST = T'S$, we have that to each $g \in L^p \cap L^{p'}$ there exists just one $f \in L^p \cap L^{p'}$ such that

$$T'(Sf - \alpha f - g) = 0.$$

Since this holds for all $T \in G_0 \cap C(L^p, L^{p'})$, we have that to each $g \in L^p \cap L^{p'}$ there exists just one $f \in L^p \cap L^{p'}$ such that

$$Sf - \alpha f - g = 0,$$

i.e. $\alpha \in \rho(S)$. This completes the proof.

A more explicit characterization of the sets $\rho(S)$ and $\sigma_P(S)$ can be obtained from Theorem 3.3 in the case in which S is a Watson transform. The following theorem shows that Theorem 3.3 contains Theorem 3.1 and 3.4 of de Snoo (1974).

Assume that $S \in I_1 \cap C(L_2, L_2)$. Then there exists a function $K \in L^\infty$ such that

$$S = M^{-1}\, m\,[K]\, MR$$

where M denotes the Mellin transform operator (on L_2) and $m[K]$ denotes multiplication by the function K (see de Snoo 1973). Write S_K (to denote dependence on K) for S. Define $K^\sim$ by $K^\sim(t) = K(-t)$, $-\infty < t < \infty$.

Theorem 3.4 — (i) $\alpha \in \rho(S_K)$ if and only if $KK^\sim - \alpha^2$ has a bounded inverse. (ii) $\alpha \in \sigma_p(S_K)$ if and only if the set of positive real t for which $K(t)\, K(-t) = \alpha^2$ has positive Lebesgue measure.

PROOF : (i) and (ii) are similarly proved : we prove (i).

By definition of $\rho(S_K)$ and Theorem 3.3, $\alpha \in \rho(S_K)$ if and only if to each $g \in L_2$ there corresponds just one $f \in L_2$ such that $S_K f - \alpha f = g$ and $S_K Tf - \alpha T'f = T'g$ for all $T \in G_0 \cap C(L_2, L_2)$. Set $T = S_K R$. Then, since R is an isometric isomorphism of L_2 onto itself, $T \in G_0 \cap C(L_2, L_2)$ and $T' = RS_K$. Hence $\alpha \in \rho(S_K)$ if and only if to each $g \in L_2$ there corresponds just one $f \in L_2$ such that

$$S_K^2 Rf - \alpha^2 Rf = RS_K g - \alpha Rg.$$

Now set $R(S_K g - \alpha g) = G$ and $Rf = F$. Then both G and $F \in L_2$, and it follows that $\alpha \in \rho(S_K)$ if and only if to each $G \in L_2$ there corresponds just one $F \in L_2$ such that $S_K^2 F - \alpha^2 F = G$. Since $S_K^2 = M^{-1} m\ [KK^{\sim}]\, M$, we see that $\alpha \in \rho(S_K)$ if and only if to each $G \in L_2$ there corresponds just one $F \in L_2$ such that $(KK^{\sim} - \alpha^2)F^{\wedge} = G^{\wedge}$, where $F^{\wedge}$ denotes the Mellin transform of F. This, by a simple argument using the properties of multiplication transforms on a σ-finite Hilbert space, holds if and only if $(KK^{\sim} - \alpha^2)$ is boundedly invertible.

In conclusion we remark that in the case in which $\alpha = 0$ the if and only if conditions of (i) and (ii) of Theorem 3.4, respectively, read: (i)′ K has a bounded inverse; (ii)′ the set of real t for which $K(t) = 0$ has positive Lebesgue measure. The mapping $S_K^2 (= T_{KK^{\sim}} \in G_0 \cap C(L_2, L_2))$ when considered as acting on the Hilbert space L_2 into itself is normal (see Duggal 1978b, p. 241). Hence it follows that a complex number α belongs to the spectrum of S_K if and only if α^2 belongs to the essential range of $KK^{\sim}$.

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