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CERTAIN SUBSPACES OF A FRECHET SPACE

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Iyer studied the space of all entire functions as a special class of sequences of complex numbers and his class has been subsequently generalised by Maddox. Further Kamthan studied the class of all entire functions having growth (1, 0) which is a subclass of the class of entire functions of exponential type. It is therefore natural to expect that some new classes can be introduced which will generalise the class introduced by Kamthan and the class of all entire functions of exponential type. Thus the purpose of this paper is to introduce and investigate these new classes.

§1. An important class of locally convex topological vector spaces consists of those which are metrizable. The class Γ of entire functions provides an interesting and useful structure on which suitable topologies can be defined to get examples of such spaces. Iyer (1948) studied the space Γ as a special class of sequences of complex numbers and showed that

$$\Gamma = \{f : f(z) = \Sigma\, a_n z^n,\ | a_n |^{1/n} \rightarrow 0 \text{ as } n \rightarrow \infty\}$$

is a nonnormable complete linear metric space with respect to the metric defined by the total paranorm g (Wilansky 1964) where

$$g(f) = \sup \{ | a_0 | ,\ | a_n |^{1/n} \ (n \geqslant 1)\}.$$

(Throughout this paper Σ denotes summation from $n = 0$ to $n = \infty$).

Subsequently Kamthan and Gupta (1974) discussed the spaces of entire functions of several complex variables. A new class of sequences was introduced by Maddox (1967, 1970) which generalises the class Γ. In fact Maddox considered

$$c_0(p) = \{a = (a_n),\ | a_n |^{p_n} \rightarrow 0 \text{ as } n \rightarrow \infty\}$$

where (p_n) is a bounded sequence of positive real numbers. Clearly if $p_n = p$ for all n, then $c_0(p) = c_0$, the set of all null sequences and if $p_n = 1/n$, then $c_0(p)$ is algebraically isomorphic to Γ. More precisely, for $f \in \Gamma$, $f(z) = \Sigma a_n z^n \leftrightarrow (a_n)$ is an algebraic isomorphism.

The subset

$$\Gamma_1 = \{f \in \Gamma : f(z) = \Sigma a_n z^n, \limsup_n n \mid a_n \mid^{1/n} < \infty\}$$

of Γ consisting of all entire functions of growth $(1, T)$, $T < \infty$, is of great importance in analysis and has drawn special attention. Kamthan (1976) studied the class

$$\Gamma_2 = \{f\colon f(z) = \Sigma a_n z^n, \ \mid n!\, a_n \mid^{1/n} \to 0 \text{ as } n \to \infty\}$$

consisting of all entire functions of growth $(1, 0)$ and showed that Γ_2 is a Frechet space with respect to the total paranorm

$$g'(f) = \sup \{ \mid a_0 \mid, \ \mid n!\, a_n \mid^{1/n} (n \geqslant 1)\}.$$

It is now natural to expect that some new classes $\Gamma_1(p)$ and $\Gamma_2(p)$ can be introduced which will generalise the classes Γ_1 and Γ_2 respectively in the same manner as $c_0(p)$ generalises the class Γ. Thus the purpose of this paper is to introduce and investigate these new concepts.

§2. Let (p_n) be a sequence of real numbers such that $p_n > 0$ and $\sup_n p_n < \infty$. We define

$$\Gamma_1(p) = \{f : f(z) = \Sigma a_n z^n, \ \limsup_n n \mid a_n \mid^{p_n} < \infty\}$$

$$\Gamma_2(p) = \{f : f(z) = \Sigma a_n z^n, \ \mid n!\, a_n \mid^{p_n} \to 0 \text{ as } n \to \infty\}.$$

Clearly $\Gamma_1(p)$ contains not only entire functions but some analytic functions also. For $p_n = \frac{1}{n^k}$, $k \geqslant 1$, $\Gamma_1(p)$ contains entire functions only. But for

$$p_n = 1 + \frac{1}{n}, \ \Gamma_1(p)$$

contains those functions which are analytic in the disc $\mid z \mid \leqslant r$, $1 \leqslant r < \infty$. However not all the entire functions belong to $\Gamma_1(p)$, e.g., $f(z) = \Sigma z^n/n^n$ does not belong to $\Gamma_1(p)$ if $p_n = \frac{1}{n^2}\cdot$ But the class $\Gamma_2(p)$ contains entire functions only. However all the entire functions do not belong to $\Gamma_2(p)$, e.g., $f(z) = e^z$. If $p_n = s + kn^{-l}$ where $l > 0$ and s and k are integers, then class $\Gamma_2(p)$ contains those entire functions whose growth are $(1, d)$, $0 \leqslant d < 1$.

Now for $f \in \Gamma_1(p)$ define

$$g_1(f) = \sup_n n \mid a_n \mid^{p_n/M}$$

and for $f \in \Gamma_2(p)$ define

$$g_2(f) = \sup_n | n!\, a_n |^{p_n/M}$$

where $M = \max(1, \sup_n p_n)$. Obviously g_1 and g_2 are paranorms and it is easy to verify that the classes $\Gamma_1(p)$ and $\Gamma_2(p)$ form complete linear metric spaces under the metric topologies generated by the paranorms g_1 and g_2 respectively.

In this paper some inclusion relations connecting the spaces $\Gamma_1(p)$ and $\Gamma_2(p)$ have been established. Further a suitable norm topology is defined over $\Gamma_2(p)$ and it is shown that the topologies generated by this norm and the paranorm g_2 are equivalent. We also obtain the characterisation of continuous linear functionals on the space $\Gamma_2(p)$. Our results include the corresponding results of Kamthan (1976), Krishnamurthy (1960a, b) and Iyer (1960).

§3. We first obtain the inclusion relation between $\Gamma_1(p)$ and $\Gamma_2(p)$. We have

Theorem 1 — If $K = \limsup_n \left(\frac{n!}{n}\right)^{p_n/M} < \infty$

and nonzero, then $\Gamma_1(p)$ is a closed subspace of $\Gamma_1(p)$.

PROOF : Clearly $\Gamma_2(p)$ is a subset of $\Gamma_1(p)$. Let $f \in \Gamma_1(p), f_i \to f$ in $\Gamma_1(p)$ and $f_i \equiv f_i(z) = \Sigma\, a_n^i\, z^n \in \Gamma_2(p)$, $i = 1, 2, \ldots$. Then given $\epsilon > 0$ there is a positive integer i_0 such that

$$n\, | a_n^i - a_n |^{p_n/M} < \frac{\epsilon}{2K} \text{ for every } i \geqslant i_0.$$

Now

$$| n!\, a_n |^{p_n/M} \leqslant | n!\, a_n^i |^{p_n/M} + | n!\, (a_n^i - a_n) |^{p_n/M}$$

$$< | n!\, a_n^i |^{p_n/M} + \left(\frac{n!}{n}\right)^{p_n/M} \cdot \frac{\epsilon}{2K}$$

$$\text{(for } i \geqslant i_0 \text{ and each } n \geqslant 1)$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ (for large } n).$$

Thus $f \in \Gamma_2(p)$ and this completes the proof.

The following theorem gives inclusion relation between $\Gamma_2(p)$ spaces.

Theorem 2 — $\Gamma_2(q) \subset \Gamma_2(p)$ if and only if

$$\liminf_n \frac{p_n}{q_n} > 0 \qquad \text{.. (3.1)}$$

PROOF : Suppose (3.1) holds and that $f \in \Gamma_2(q)$. Then there is $\alpha > 0$ such that $p_n > \alpha q_n$ for large n. Hence for large n,

$$| n!\, a_n |^{p_n} \leqslant (| n!\, a_n |^{q_n})^{\alpha}$$

since $| n!\, a_n | \leqslant 1$ for such n. Hence $f \in \Gamma_2(p)$.

Conversely suppose that the inclusion holds, but (3.1) is false. Then we can determine an increasing sequence $n_1 < n_2 < \ldots$ such that $p_{ni} < \frac{1}{i}\, q_{ni} \ldots$. Define

$$n!\, a_n = \begin{cases} (i^{-1})^{1/q_n} & (n = n_i) \\ 0 & (n \neq n_i) \end{cases}$$

Then $f \in \Gamma_2(q)$ but

$$| n_i a_{ni} |^{p_{ni}} > \exp(-\log_i/i) > e^{-1/2}$$

which contradicts the fact that $f \in \Gamma_2(p)$. This completes the proof.

§4. We now define a norm on $\Gamma_2(p)$ and show that the norm topology is equivalent to the paranorm topology. For $f \equiv f(z) = \Sigma\, a_n z^n \in \Gamma_2(p)$ and for any $\delta > 0$ define

$$\| f; \delta \| = \sup_n \frac{n!\, | a_n |}{\delta^{M/p_n}}. \quad \ldots(4.1)$$

Clearly for each $\delta > 0$, (4.1) defines a norm on $\Gamma_2(p)$. Denote the corresponding normed space by $\Gamma_2(p, \delta)$. Note that as δ decreases, norm increases; so topology becomes stronger. Let $\Gamma_2(p)$ be the weakest topology which is stronger than each $\Gamma_2(p, \delta)$. Obviously $\Gamma_2(p)$ is generated by the family $\{\Gamma_2(p, \delta); \delta > 0\}$. Further it can be shown that $\Gamma_2(p)$ is a Frechet space under the induced metric

$$d(f, g) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\| f - g; (1/i) \|}{1 + \| f - g; (1/i) \|}.$$

The following lemma can be proved on the lines adopted by Krishnamurthy (1960) and therefore we state it without proof.

Lemma 1 — If $d(f, 0) \geqslant k$ $(0 < k < 2)$, then $\| f; \delta \| \geqslant \frac{k}{2 - k}$ for some $\delta = \delta_0$ where $0 < \delta_0 \leqslant 1$, and therefore for all values of $\delta \leqslant \delta_0$.

It follows from Lemma 1 that if a series converges in $\Gamma_2(p, \delta)$ for each $\delta > 0$, then it converges in $\Gamma_2(p)$. Converse is obviously true.

Theorem 3 — Let $\Gamma_2(p)$ be the topology generated by the norm (4.1) and let T be the usual topology generated by the paranorm g_2. Then the two topologies $\Gamma_2(p)$ and T are equivalent.

PROOF : Let $f_i \to 0$ in T as $i \to \infty$. Let $\delta > 0$ and $0 < \alpha < 1$ be given. Choose $\epsilon > 0$ such that $\epsilon < \delta\alpha$. Then $\frac{\epsilon}{\delta} < \alpha < 1$. So, for $n \geqslant 1$

$$\left(\frac{\epsilon}{\delta}\right)^{M/p_n} < \alpha^{M/p_n} \leqslant \alpha. \qquad \ldots(4.2)$$

Since $f_i \to 0$ in T, for $\epsilon > 0$ chosen above, there exists i_0 such that for $i \geqslant i_0$,

$$| n!\, a_n^i |^{p_n/M} < \epsilon \text{ for each } n \geqslant 1.$$

Therefore for $i \geqslant i_0$

$$\frac{| n!\, a_n^i |}{\delta^{M/p_n}} < \left(\frac{\epsilon}{\delta}\right)^{M/p_n} \leqslant \alpha$$

by (4.2) and thus for $i \geqslant i_0$,

$$\| f_i ; \delta \| \leqslant \alpha \qquad \ldots(4.3)$$

which shows that $f_i \to 0$ in $\Gamma_2(p, \delta)$ for every $\delta > 0$ and so in $\Gamma_2(p)$. [In the above there is no loss of generality in assuming that $\alpha < 1$; since if $\alpha > 1$, we can choose an α' such that $\alpha' < 1 < \alpha$ and repeat the above arguments with α' inplace of α and ultimately reach (4.3)].

Conversely suppose $f_i \to 0$ in $\Gamma_2(p)$. Then $f_i \to 0$ in $\Gamma_2(p, \delta)$ for every $\delta > 0$. Given $\alpha > 0$, choose ϵ, $\delta > 0$ such that $\delta < \alpha$ and $\epsilon < \frac{\alpha}{\delta}$. Then, for every $n \geqslant 1$,

$$\epsilon < \frac{\alpha}{\delta} \leqslant \left(\frac{\alpha}{\delta}\right)^{M/p_n}$$

and so

$$\delta\, \epsilon^{p_n/M} \leqslant \alpha. \qquad \ldots(4.4)$$

Since $f_i \to 0$ in $\Gamma_2(p, \delta)$ for every $\delta > 0$, for ϵ, $\delta > 0$ chosen above, we can find i_0, such that for $i \geqslant i_0$,

$$\frac{| n!\, a_n^i |}{\delta^{M/p_n}} < \epsilon \text{ for every } n \geqslant 1.$$

So for every $n \geqslant 1$,

$$| n!\, a_n^i |^{p_n/M} < \delta\, \epsilon^{p_n/M} \leqslant \alpha$$

by (4.4). Thus given $\alpha > 0$, we get for $i \geqslant i_0$,

$$g_2(f_i) \leqslant \alpha$$

which implies that $f_i \to 0$ in T as $i \to \infty$. This completes the proof of the theorem.

§5. We now state a theorem which characterises the continuous linear functionals on $\Gamma_2(p)$. We omit the proof since it uses ideas similar to those used in Iyer (1960) and Kamthan and Gupta (1974).

Theorem 4 — (a) Every continuous linear functional in $\Gamma_2^*(p, \delta)$ is of the form

$$\psi(f) = \Sigma\, c_n a_n,\ f(z) = \Sigma\, a_n z^n, \text{ where}$$

$$\Sigma \frac{|c_n|}{n!}\, \delta^{M/p_n} < \infty.$$

(b) Every continuous linear functional in $\Gamma_2^*(p)$ is of the form

$$\psi(f) = \Sigma\, c_n a_n,\ f(z) = \Sigma\, a_n z^n, \text{ where}$$

$$\limsup_n \left(\frac{|c_n|}{\left(\frac{n}{\rho}\right)!} \right)^{p_n/M} < \infty \text{ where } \rho \leqslant 1.$$

We conclude the paper by stating a result of Γ_2. The proof uses ideas similar to those used to prove the corresponding Theorem for Γ (see Goffman and Pedrick 1974, Chap. 5).

Theorem 5 — A sequence $\{f_i\}$ converges to f on Γ_2 if and only if it converges uniformly to f on every compact set.

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