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ON SOME INTEGRAL RELATIONS INVOLVING THE GENERAL *H*-FUNCTION OF SEVERAL COMPLEX VARIABLES

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In this paper we establish two integral relations for the multivariable *H*-function of k complex variables due to Srivastava and Panda (1976a). Examples are constructed to illustrate the results.

1. INTRODUCTION

Recently, many authors have given double integrals [see Dahiya and Singh (1971), Prasad and Ram (1973), and Singh (1977)]. Our aim is to obtain integral relations involving the general *H*-function of k complex variables due to Srivastava and Panda (1976a). These results contain those of earlier workers as special cases.

The general *H*-function due to Srivastava and Panda (1976a) is defined by [Srivastava and Panda 1976a, p. 271, eqn. (4.1)] :

$$H^{O,N:(M',N');\ldots;(M^{(k)},N^{(k)})}_{A,B:[P',Q'];\ldots;[P^{(k)},Q^{(k)}]}\left(\begin{matrix}[(a):A^{(1)},\ldots,A^{(k)}]:[(p'):\alpha'];\ldots;[(p^{(k)}):\alpha^{(k)}];\\ [(b):B^{(1)},\ldots,B^{(k)}]:[(q'):\beta'];\ldots;[(q^{(k)}):\beta^{(k)}];\end{matrix}\; x_1,\ldots,x_k\right)$$

$$=\frac{1}{(2\pi i)^k}\int_{L_1}\cdots\int_{L_k}\phi(s_1,\ldots,s_k)\prod_{u=1}^{k}\left\{\theta_u(s_u)\,x_u^{s_u}\,ds_u\right\} \qquad \ldots(1.1)$$

where
$$\phi(s_1,\ldots,s_k)=\frac{\prod_{j=1}^{N}\Gamma(1-a_j+\sum_{u=1}^{k}A_j^{(u)}s_u)}{\prod_{j=N+1}^{A}\Gamma(a_j-\sum_{u=1}^{k}A_j^{(u)}s_u)\prod_{j=1}^{B}\Gamma(1-b_j+\sum_{u=1}^{k}B_j^{(u)}s_u)} \qquad \ldots(1.2)$$

and
$$\theta_u(s_u)=\frac{\prod_{j=1}^{M^{(u)}}\Gamma(q_j^{(u)}-\beta_j^{(u)}s_u)\prod_{j=1}^{N^{(u)}}\Gamma(1-p_j^{(u)}+\alpha_j^{(u)}s_u)}{\prod_{j=M^{(u)}+1}^{Q^{(u)}}\Gamma(1-q_j^{(u)}+\beta_j^{(u)}s_u)\prod_{j=N^{(u)}+1}^{P^{(u)}}\Gamma(p_j^{(u)}-\alpha_j^{(u)}s_u)}$$

$$(u=1,\ldots,k), \qquad \ldots(1.3)$$

an empty product is interpreted as 1, the coefficients $A_j^{(u)}, j = 1, \ldots, A$; $B_j^{(u)}$, $j = 1, \ldots, B$, $\alpha_j^{(u)}$, $j = 1, \ldots, P^{(u)}$; $\beta_j^{(u)}$, $j = 1, \ldots, Q^{(u)}$; and $u = 1, \ldots, k$, are positive numbers. Further, N, A, B, $M^{(u)}$, $N^{(u)}$, $P^{(u)}$, $Q^{(u)}$ are integers such that $0 \leqslant N \leqslant A$, $0 \leqslant M^{(u)} \leqslant Q^{(u)}$, $B \geqslant 0$ and $0 \leqslant N^{(u)} \leqslant P^{(u)}$, $u = 1, \ldots, k$. The contour L_u in the complex s_u-plane is of the Mellin-Barnes type running from $-i\infty$ to $i\infty$ with indentations, if necessary, to ensure that it separates one set of poles of the integrand from the other. The various parameters are so restricted, that the poles are all simple, and with the points $x_u = 0$, $u = 1, \ldots, k$, being tacitly excluded, the multiple integral (1.1) converges absolutely if

$$|\arg(x_u)| < \tfrac{1}{2}\pi V_u \qquad \ldots(1.4)$$

where

$$V_u = -\sum_{j=N+1}^{A} A_j^{(u)} + \sum_{j=1}^{N^{(u)}} \alpha_j^{(u)} - \sum_{j=N^{(u)}+1}^{P^{(u)}} \alpha_j^{(u)} - \sum_{j=1}^{B} B_j^{(u)} + \sum_{j=1}^{M^{(u)}} \beta_j^{(u)} - \sum_{j=M^{(u)}+1}^{Q^{(u)}} \beta_j^{(u)} > 0 \; (u = 1, \ldots, k). \qquad \ldots(1.5)$$

The left-hand side of (1.1) would be written as

$$H^{O,N:((\;))}_{A,B:([\;])}\left[\begin{matrix} x_1 \\ \vdots \\ x_k \end{matrix}\left|\begin{matrix} [(a) : A^{(1)}, \ldots, A^{(k)}] : ([\quad]) \\ \\ [(b) : B^{(1)}, \ldots, B^{(k)}] : ([\quad]) \end{matrix}\right.\right]$$

or $$H\begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}$$

if the parameters of the general H-function are exactly as given in (1.1).

From Srivastava and Panda (1976b, p. 131) we have

$$H\begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} = \begin{cases} O(|x_1|^{\lambda_1} \ldots |x_k|^{\lambda_k}), \max\{|x_1|, \ldots, |x_k|\} \to 0, \\ O(|x_1|^{-M_1} \ldots |x_k|^{-M_k}), N \equiv 0, \min\{|x_1|, \ldots, |x_k|\} \to \infty, \end{cases} \qquad \ldots(1.6)$$

where, with $u = 1, \ldots, k$,

$$\lambda_j = \frac{q_j^{(u)}}{\beta_j^{(u)}} \; ; \; j = 1, \ldots, M^{(u)},$$

$$M_j = \frac{1 - p_j^{(u)}}{\alpha_j^{(u)}} \; ; \; j = 1, \ldots, N^{(u)}.$$

Notations and Known Results

The following known results (see Sneddon 1956) will be required hereafter:

$$\int_0^{\pi/2} \cos(2w\theta)(\sin\theta)^v\, d\theta = \frac{\Gamma(v+1)\,\Gamma(\frac{1}{2}+w)\,\Gamma(\frac{1}{2}-w)}{2^{v+1}\,\Gamma(\frac{1}{2}v+w+1)\,\Gamma(\frac{1}{2}v-w+1)} \quad \text{...(1.7)}$$

$$\int_0^{\pi/2} \cos(2w\theta)(\cos\theta)^v\, d\theta = \frac{\pi\Gamma(v+1)}{2^{v+1}\Gamma(\frac{1}{2}v+w+1)\,\Gamma(\frac{1}{2}v-w+1)} \quad \text{...(1.8)}$$

where $R(v) > 0$ and w is any integer.

For the sake of brevity, we use the following notations:

(i) $((M^{(k)}, N^{(k)}))$ for $(M', N'); \ldots; (M^{(k)}, N^{(k)})$

(ii) $([P^{(k)}, Q^{(k)}])$ for $[P', Q']; \ldots; [P^{(k)}, Q^{(k)}]$

(iii) $\theta_{x,y}$ for $\cos\left\{2w\left(\tan^{-v}\frac{y}{x}\right)\right\}(x^2+y^2)^{-v}$

(iv) $([(p^{(k)}):\alpha^{(k)}])$ for $[(p'):\alpha']; \ldots; [(p^{(k)}):\alpha^{(k)}]$

(v) $([(q^{(k)}):\beta^{(k)}])$ for $[(q'):\beta']; \ldots; [(q^{(k)}):\beta^{(k)}]$

(vi) $\Gamma(a \pm b)$ for $\Gamma(a+b)\cdot\Gamma(a-b)$

(vii) (δ_k) for $\delta_1, \ldots, \delta_k$

(viii) $((t_P, T_P)) = (t_j, T_j)_{1,P}$ for $(t_1, T_1), \ldots, (t_P, T_P)$.

Throughout the present paper (a) denotes the sequence of A parameters $a_1, a_2, \ldots, a_A$, while $(p^{(u)})$ and $(q^{(u)})$ stand for the sequences

$$p_j^{(u)}, j = 1, \ldots, P^{(u)} \text{ and } q_j^{(u)}, j = 1, \ldots, Q^{(u)},$$

respectively. The conditions of existence for the H-functions appearing in this paper, as well as suitable restrictions on the parameters, are assumed as given by Srivastava and Panda (1976a).

2. The Main Integrals

If $R(v) > 0$, w is an integer, $0 \leqslant N \leqslant A$, $B \geqslant 0$, $0 \leqslant M^{(u)} \leqslant Q^{(u)}$, $0 \leqslant N^{(u)} \leqslant P^{(u)}$, $\delta_u > 0$, $u = 1, \ldots, k$, and appropriate modified conditions corresponding to conditions (1.4) and (1.5) for the existence of the multivariate H-function are satisfied then, *Integral I*

$$\int_0^{\pi/2} \cos(2w\theta)(\sin\theta)^{2v}\, H^{O,N:((\ ))}_{A,B:([\ ])}\left[\begin{array}{c|c} z_1 z^{2\delta_1}(\sin\theta)^{2\sigma_1} & [(a):A^{(1)},\ldots,A^{(k)}]:([\ ]) \\ \vdots & \\ z_k z^{2\delta_k}(\sin\theta)^{2\sigma_k} & [(b):B^{(1)},\ldots,B^{(k)}]:([\ ]) \end{array}\right] d\theta$$

$$= \frac{\Gamma(\frac{1}{2}\pm w)}{2\sqrt{\pi}}\, H^{O,N+2:((\ ))}_{A+2,B+2:([\ ])}\left[\begin{array}{c|c} z_1 z^{2\delta_1} & \phi_1 \\ \vdots & \\ z_k z^{2\delta_k} & \phi_2 \end{array}\right] \quad \ldots(2.1)$$

where ϕ_1 and ϕ_2 denote the parameters,

$$[-v;(\sigma_k)]\,[\tfrac{1}{2}-v;(\sigma_k)],\,[(a):A^{(1)},\ldots,A^{(k)}]:[(p^{(k)}),\alpha^{(k)}]$$

and

$$[(b):B^{(1)},\ldots,B^{(k)}],\,[-v\pm w:(\sigma_k)]:([(q^{(k)}):\beta^{(k)}]),$$

respectively.

Integral II

If the conditions given in Integral I are assumed to hold, then

$$\int_0^{\pi/2} \cos(2w\theta)(\cos\theta)^{2v}$$

$$H^{O,N:((\ ))}_{A,B:([\ ])}\left[\begin{array}{c|c} z_1 z^{2\delta_1}(\cos\theta)^{2\sigma_1} & [(a):A^{(1)},\ldots,A^{(k)}]:([\ ]) \\ \vdots & \\ z_k z^{2\delta_k}(\cos\theta)^{2\sigma_k} & [(b):B^{(1)},\ldots,B^{(k)}]:([\ ]) \end{array}\right] d\theta$$

$$= \frac{\sqrt{\pi}}{2}\, H^{O,N+2:((\ ))}_{A+2,B+2:([\ ])}\left[\begin{array}{c|c} z_1 z^{2\delta_1} & \phi_1 \\ \vdots & \\ z_k z^{2\delta_k} & \phi_2 \end{array}\right]. \quad \ldots(2.2)$$

Proofs of (2.1) *and* (2.2)

To prove (2.1), express the general H-function appearing on the left-hand side of (2.1) in contour integral form by (1.1), change the order of integrals, use result (1.7) and the duplication formula for Gamma function (Erdélyi *et al.* 1953, p. 5) and then interpret by means of (1.1); the r.h.s. of (2.1) follows immediately.

The proof of (2.2) can be developed in a similar way, but we use (1.8) in place of (1.7).

3. Integral Relations

The following two integral relations are proved in this section.

Relation I

$$\int_0^\infty \int_0^\infty \theta_{x,y}\, y^{2v}\, H^{O,N:((\ ))}_{A,B:([\ ])}\left[\begin{array}{c} z_1 y^{2\sigma_1}(x^2+y^2)^{2\delta_1-\sigma_1} \\ \vdots \\ z_k y^{2\sigma_k}(x^2+y^2)^{2\delta_k-\sigma_k} \end{array}\middle| \begin{array}{l} [(a):A^{(1)},\ldots,A^{(k)}]:([\ ]) \\ [(b):B^{(1)},\ldots,B^{(k)}]:([\ ]) \end{array}\right] f(x^2+y^2)\,dx\,dy$$

$$= \frac{(\frac{1}{2}\pm w)}{4\sqrt{\pi}} \int_0^\infty H^{O,N+2:((\ ))}_{A+2,B+2:([\ ])}\left[\begin{array}{c} z_1 z^{2\delta_1} \\ \vdots \\ z_k z^{2\delta_k}\end{array}\middle|\begin{array}{c}\phi_1 \\ \\ \phi_2\end{array}\right] f(z)\,dz \qquad \ldots(3.1)$$

Relation II

$$\int_0^\infty \int_0^\infty \theta_{x,y}\, x^{2v}\, H^{O,N:((\ ))}_{A,B:([\ ])}\left[\begin{array}{c} z_1 x^{2\sigma_1}(x^2+y^2)^{2\delta_1-\sigma_1} \\ \vdots \\ z_k x^{2\sigma_k}(x^2+y^2)^{2\delta_k-\sigma_k} \end{array}\middle| \begin{array}{l} [(a):A^{(1)},\ldots,A^{(k)}]:([\ ]) \\ [(b):B^{(1)},\ldots,B^{(k)}]:([\ ]) \end{array}\right] f(x^2+y^2)\,dx\,dy$$

$$= \frac{\sqrt{\pi}}{4} \int_0^\infty H^{O,N+2:((\ ))}_{2+A,B+2:([\ ])}\left[\begin{array}{c} z_1 z^{2\delta_1} \\ \vdots \\ z_k z^{2\delta_k}\end{array}\middle|\begin{array}{c}\phi_1 \\ \\ \phi_2\end{array}\right] f(z)\,dz \qquad \ldots(3.2)$$

provided that $R(v) > 0$, w is any integer,

$$0 \leqslant N \leqslant A,\ B \geqslant 0,\ 0 \leqslant M^{(u)} \leqslant Q^{(u)},\ 0 \leqslant N^{(u)} \leqslant P^{(u)},$$

$0 < \frac{\sigma_u}{2} < \delta_u$, $u = 1, \ldots, k$, and appropriate modified conditions corresponding to (1.4) and (1.5) for the existence of the general H-function are satisfied.

Proof of (3.1)

Put $z = r^2$ in (2.1), multiply both sides by $rf(r^2)$ and integrate with respect to r between $(0, \infty)$. Then on writing $x = r\cos\theta$, $y = r\sin\theta$ and simplifying the integral relation (3.1) is established.

Proof of (3.2)

Proceed as in the proof of (3.1) and use result (2.2) in place of (2.1).

4. Examples

By putting

$$f(z) = z^{\sigma-1}\ H^{M,O}_{P,Q}\left[\frac{az^2}{4}\middle|\begin{array}{l}((g_P, G_P)) \\ ((h_Q, H_Q))\end{array}\right]$$

in (3.1) and (3.2), using the definition (1.1) and a known result due to Mittal and Gupta [1970, p. 145], we obtain the following results:

Example 1

$$\int_0^\infty \int_0^\infty \theta_{x,v}\, y^{2v}(x^2+y^2)^{\sigma-1}\, H_{P,Q}^{M,O}\left[\frac{a(x^2+y^2)}{4}\,\middle|\,\begin{matrix}((g_P, G_P))\\((h_Q, H_Q))\end{matrix}\right]$$

$$\times\, H^{O,N:((\ ))}_{A,B:([\ ])}\left[\begin{matrix} z_1 y^{2\sigma_1}(x^2+y^2)^{2\delta_1-\sigma_1}\\ \vdots \\ z_k y^{2\sigma_k}(x^2+y^2)^{2\delta_k-\sigma_k}\end{matrix}\,\middle|\,\begin{matrix}[(a):A^{(1)},\ldots A^{(k)}]:([\ ])\\ \\ [(b):B^{(1)},\ldots B^{(k)}]:([\ ])\end{matrix}\right]dx\,dy$$

$$=\frac{2^{\sigma-3}\Gamma(\frac{1}{2}\pm w)}{\pi a^{\sigma}}\, H^{O,N+M+2:((\ ))}_{A+Q+2,B+P+2:([\ ])}\left[\begin{matrix} z_1\left(\frac{4}{a}\right)^{\delta_1}\\ \vdots \\ z_k\left(\frac{4}{a}\right)^{\delta_k}\end{matrix}\,\middle|\,\begin{matrix}\phi_3\\ \\ \phi_4\end{matrix}\right] \qquad \ldots(4.1)$$

provided that

(i) conditions mentioned along with (3.1) and (3.2) are satisfied,

(ii) $v_1=\sum_{j=1}^{M} H_j-\sum_{j=M+1}^{Q} H_j-\sum_{j=1}^{P} G_j>0$,

(iii) $|\arg a|<\frac{1}{2}v_1\pi$,

(iv) $\sum_{j=1}^{Q} H_j-\sum_{j=1}^{P} G_j \geqslant 0$

ϕ_3 and ϕ_4 denote the parameters

$$\left[1-h_j-\frac{\sigma}{2}H_j:H_j(\delta_k)\right]_{1,Q},\ [-v:(\sigma_k)],\ [\tfrac{1}{2}-v:(\sigma_k)],$$
$$[(a):A^{(1)},\ldots,A^{(k)}]:[(p'):\alpha'];\ \ldots;\ [(p^{(k)}):\alpha^{(k)}],$$

and

$$[(b):B^{(1)},\ldots,B^{(k)}]:[-v\pm w:(\sigma_k)],$$
$$\left[1-g_j-\frac{\sigma}{2}G_j:G_j(\delta_k)\right]_{1,P}:\ [(q'):\beta'];\ \ldots;\ [(q^{(k)}):\beta^{(k)}],$$ respectively.

Conditions mentioned in Example 1 are assumed to hold.

Example 2

$$\int_0^\infty \int_0^\infty \theta_{x,v}\, x^{2v}(x^2+y^2)^{\sigma-1}\, H_{P,Q}^{M,O}\left[\frac{a(x^2+y^2)}{4}\,\middle|\,\begin{matrix}((g_P, G_P))\\((h_Q, H_Q))\end{matrix}\right]\times$$

(*equation continued on p.* 983)

$$H^{O,N:((\ ))}_{A,B:([\ ])}\left[\begin{matrix} z_1 x^{2\sigma_1}(x^2+y^2)^{2\delta_1-\sigma_1} \\ \vdots \\ z_k x^{2\sigma_k}(x^2+y^2)^{2\delta_k-\sigma_k}\end{matrix}\middle|\begin{matrix}[(a):A^{(1)},\ldots,A^{(k)}]:([\ ]) \\ \\ [(b):B^{(1)},\ldots B^{(k)}:([\ ])\end{matrix}\right] dx\,dy$$

$$= \frac{2^{\sigma-3}}{a^{\sigma}} H^{O,N+M+2\ ((\ ))}_{A+Q+2,B+P+2:([\ ])}\left[\begin{matrix} z_1\left(\frac{4}{a}\right)^{\delta_1} \\ \vdots \\ z_k\left(\frac{4}{a}\right)^{\delta_k}\end{matrix}\middle|\begin{matrix}\phi_3 \\ \\ \phi_4\end{matrix}\right] \quad \ldots(4.2)$$

5. Particular Cases

By putting $k=2$, $\sigma_1=\sigma_2=1$, $\delta_1=\delta_2=\frac{\rho}{2}$ in (3.1) and (3.2), setting all the A's, B's, α's, β's equal to unity and replacing b_j by $1-b_j$ and making some minor changes, the known results due to Singh (1977, p. 190) are obtained.

Again, if we write in (4.1) and (4.2), $k=2$, $M=Q=4$, $P=2$, $g_1=1+\lambda$, $g_2=1-\lambda$, $h_1=\frac{1}{2}$, $h_2=1$, $h_3=1+\mu$, $h_4=1-\mu$,

$$G_1=G_2=H_1=H_2=H_3=H_4=\delta_1=\delta_2=\sigma_1=\sigma_2=a=1,$$

replace b_j by $1-b_j$ and set all of the A's, B's, α's and β's equal to unity and make some minor changes, we arrive at known results also due to Singh (1977, p. 194). The multivariate H-function due to Srivastava and Panda (1976a) includes almost all functions of one and more variables defined so far; therefore, by specializing the parameters therein, we can obtain a large number of new and known integral relations.

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References

Dahiya, R. S., and Singh, B. (1971). On Meijer's G-function. *Proc. Indian Acad. Sci.*, **74**, 167–71.

Erdélyi, A. *et. al.* (1953). Higher Transcendental Functions, Vol. 1. McGraw-Hill Book Co., Inc., New York.

Prasad, Y. N., and Ram, S. D. (1973). On some double integrals involving Fox's H-function. *Prog. Math. (Allahabad)*, **7**, 13–20.

Singh, V. N. (1977). On some double integrals involving Meijer's G-function. *Indian J. pure appl. Math.*, **8**, 189–95.

Sneddon, I. N. (1956). Special Functions of Mathematical Physics and Chemistry. Interscience, New York.

Srivastava, H. M., and Panda, R. (1976a). Some bilateral generating functions for a class of generalized hypergeometric polynomials. *J. Reine angew. Math.*, **283/284**, 265–74.

————(1976b). Expansion theorems for the H-function of several complex variables. *J. Reine angew. Math.*, **288**, 129–45.