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CONTIGUOUS RELATIONS FOR THE GENERAL *H*-FUNCTION OF SEVERAL VARIABLES

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Applying the technique used by Buschman (1972), we give some basic contiguous relations for the general multivariable *H*-function due to Srivastava and Panda (1976). Some similar relations have been arrived at by Tandon (1980a), employing the technique of Srivastava (1966). Tandon (1980b) has also given some finite summation formulas for the general multivariable *H*-function introduced by Srivastava and Panda (1976).

1. INTRODUCTION

Tandon (1980a, b) in two papers has studied the contiguous relations, and finite summation formulae, involving the multivariable *H*-function due to Srivastava and Panda (1976). The technique used by Tandon is that of Srivastava (1966). In the present paper we have obatined contiguous relations for the general *H*-function of k variables due to Srivastava and Panda (1976), by using the method developed by Buschman (1972) who obtained the contiguous relations of the *H*-function of Fox (1961). The *H*-function of several variables is defined by the multiple integral [Srivastava and Panda 1976a, p. 271, eqn. (4.1) *et seq.*] :

$$H^{O,N:M^{(1)},N^{(1)};\ldots;M^{(k)},N^{(k)}}_{A,B:P^{(1)},Q^{(1)};\cdots;P^{(k)},Q^{(k)}}$$

$$\left(\begin{matrix}[(a):A^{(1)},\ldots,A^{(k)}]:[(p');\alpha'];\ldots;[(p^{(k)}):\alpha^{(k)}]; \\ [(b):B^{(1)},\ldots,B^{(k)}]:[(q');\beta'];\ldots;[(q^{(k)});\beta'^{(k)}];\end{matrix}\ x_1,\ldots,x_k\right)$$

$$=\frac{1}{(2\pi i)^k}\int_{L_1}\cdots\int_{L_k}\phi(s_1,\ldots,s_k)\,\theta_1(s_1)\ldots\theta_k(s_k)\;x_1^{s_1}\ldots x_k^{s_k}\,ds_1\ldots ds_k, \quad \ldots(1.1)$$

where

$$\phi(s_1,\ldots,s_k)=\frac{\prod_{j=1}^{N}\Gamma(1-a_j+\prod_{u=1}^{k}A_j^{(u)}s_u)}{\prod_{j=N+1}^{P}\Gamma(a_j-\sum_{u=1}^{k}A_j^{(u)}s_u)\prod_{j=1}^{Q}\Gamma(1-b_j+\sum_{u=1}^{k}B_j^{(u)}s_u)} \quad \ldots(1.2)$$

and
$$\theta_u(s_u) = \frac{\prod_{j=1}^{M^{(u)}} \Gamma(d_j^{(u)} - \beta_j^{(u)} s_u) \prod_{j=1}^{N^{(u)}} \Gamma(1 - c_j^{(u)} + \alpha_j^{(u)} s_u)}{\prod_{j=M^{(u)}+1}^{Q^{(u)}} \Gamma(1 - d_j^{(u)} + \beta_j^{(u)} s_u) \prod_{j=N^{(u)}+1}^{P^{(u)}} \Gamma(c_j^{(u)} - \alpha_j^{(u)} s_u)}, \quad \ldots(1.3)$$

$$u = 1, \ldots, k.$$

For various restrictions on the parameters, and conditions of convergence regarding (1.1), the reader may go through the original papers of Srivastava and Panda (1976a, b). Here $L_1, \ldots, L_k$ are suitable contours, and the points $x_u = 0$, $u = 1, \ldots, k$ are tacitly excluded. Throughout the present paper appropriate conditions of convergence, and suitable restrictions on the parameters, are assumed.

Notations

In this paper (a) denotes the sequence of A parameters $a_1, a_2, \ldots, a_A$. $(p^{(u)})$ and $(q^{(u)})$ denote the sequences, $c_j^{(u)}, j = 1, \ldots, P^{(u)}$ and $d_j^{(u)}, j = 1, \ldots, Q^{(u)}$ respectively, $u = 1, \ldots, k$. For the sake of brevity, we write

$$W_u = \frac{W}{\alpha_1^{(u)}}, \quad u = 1, \ldots, k,$$

where W denotes the product $\alpha_1^{(1)} \cdot \alpha_1^{(2)} \ldots \alpha_1^{(k)}$.

Further $H[d_1^{(1)} + 1]$ represents the multiple contour integral (1.1) in which $d_1^{(1)}$ is replaced by $d_1^{(1)} + 1$, and all other parameters remain unaltered, and so on. The general H-function of k variables due to Srivastava and Panda (1976a) is denoted by the symbol H.

2. The Main Results

We prove the following recurrence relations:

$$\alpha_{p(u)}^{(u)} H[d_1^{(u)} + 1] - \beta_1^{(u)} H[c_{p(u)}^{(u)} - 1] = H \begin{vmatrix} d_1^{(u)} & c_{p(u)}^{(u)} - 1 \\ \beta_1^{(u)} & \alpha_{p(u)}^{(u)} \end{vmatrix}, \quad \ldots(2.1)$$

$$M^{(u)} \geqslant 1, P^{(u)} > N^{(u)}, u = 1, \ldots, k$$

$$\alpha_{p(u)}^{(u)} H[c_1^{(u)} - 1] + \alpha_1^{(u)} H[c_{p(u)}^{(u)} - 1] = -H \begin{vmatrix} c_1^{(u)} - 1 & c_{p(u)}^{(u)} - 1 \\ \alpha_1^{(u)} & \alpha_{p(u)}^{(u)} \end{vmatrix}, \quad \ldots(2.2)$$

$$P^{(u)} > N^{(u)} \geqslant 1, u = 1, \ldots, k$$

$$\beta_{Q(u)}^{(u)} H[c_1^{(u)}-1] - \alpha_1^{(u)} H[d_{Q(u)}^{(u)}+1] = -H \begin{vmatrix} c_1^{(u)}-1 & d_{Q(u)}^{(u)} \\ \alpha_1^{(u)} & \beta_{Q(u)}^{(u)} \end{vmatrix}, \quad \ldots(2.3)$$

$$Q^{(u)} > M^{(u)},\ N^{(u)} \geqslant 1,\ u = 1, \ldots, k.$$

Derivation of the Recurrence Relations (2.1), (2.2) *and* (2.3)

In this section we give the proof of (2.1) only, as the proofs of (2.2) and (2.3) are similarly obtained. We observe that by using $\Gamma(z+1) = z\,\Gamma(z)$, after replacing $d_1^{(u)}$ by $d_1^{(u)}+1$, on the right-hand side of (1.1), introduce the extra multiplying factors, $d_1^{(u)} - \beta_1^{(u)} s_u$, $u = 1, \ldots, k$. Further the replacement of $c_{P(u)}^{(u)}$ by $c_{P(u)}^{(u)} - 1$ introduces the extra multiplying factors $c_{P(u)}^{(u)} - 1 - \alpha_{P(u)}^{(u)} s_u$, $u = 1, \ldots, k$.

Thus, we form, the recurrence formula

$$LH[d_1^{(u)}+1] + MH[c_{P(u)}^{(u)}-1] = R \cdot H,$$

involving the undetermined coefficients L, M and R, and then require that,

$$L(d_1^{(u)} - \beta_1^{(u)} s_u) + M(c_{P(u)}^{(u)} - 1 - \alpha_{P(u)}^{(u)} s_u) = R,$$

be an identity in s_u, $u = 1, \ldots, k$.

By evaluating and substituting the values of L, M and R the required recurrence formula (2.1) is established.

Proofs of (2.2) and (2.3) are derived in a similar way. For this purpose we observe that the replacement of $c_1^{(u)}$ by $c_1^{(u)} - 1$, on the right-hand side of (1.1) results into the introduction of additional multiplying factors, $1 - c_1^{(u)} - \alpha_1^{(u)} s_u$, $u = 1, \ldots, k$. In the same way if we write $d_{Q(u)}^{(u)} + 1$ for $d_{Q(u)}^{(u)}$, extra factors of the type,

$$-\,d_{Q(u)}^{(u)} + \beta_{Q(u)}^{(u)} s_u,\ u = 1, \ldots, k,$$

are introduced on the right-hand side of the multiple integral (1.1). Each of the contiguous relation gives rise to k recurrence relations for each u ($u = 1, \ldots, k$).

3. Further Results

Now dealing with $\phi(s_1, \ldots, s_k)$ we obtain the following set of recurrence relations:

$$WH[a_1 - 1] - W_1 A_1^{(1)} H[c_1^{(1)} - 1] - \ldots - W_k A_1^{(k)} H[c_1^{(k)} - 1]$$

$$= H \begin{vmatrix} A_1^{(1)} & 0 & \ldots & \ldots & \ldots & 0 & \alpha_1^{(1)} \\ A_1^{(2)} & 0 & \ldots & \ldots & \ldots & \alpha_1^{(2)} & 0 \\ \vdots & \vdots & & & & \vdots & \vdots \\ A_1^{(k)} & \alpha_1^{(k)} & \ldots & \ldots & \ldots & 0 & 0 \\ a_1 - 1 & c_1^{(k)} - 1 & \ldots & \ldots & \ldots & c_1^{(2)} - 1 & c_1^{(1)} - 1 \end{vmatrix} \quad \ldots(3.1)$$

provided that $N \geqslant 1$ and $N^{(u)} \geqslant 1$, $u = 1, \ldots, k$.

$$WH[a_P - 1] + A_P^{(1)} W_1 H[c_1^{(1)} - 1] + \ldots + A_P^{(k)} W_k H[c_1^{(k)} - 1]$$

$$= -H \begin{vmatrix} A_P^{(1)} & 0 & \ldots & \ldots & \ldots & 0 & \alpha_1^{(1)} \\ A_P^{(2)} & 0 & \ldots & \ldots & \ldots & \alpha_1^{(2)} & 0 \\ \vdots & \vdots & & & & \vdots & \vdots \\ A_P^{(k)} & \alpha_1^{(k)} & \ldots & \ldots & \ldots & 0 & 0 \\ a_P - 1 & c_1^{(k)} - 1 & \ldots & \ldots & \ldots & c_1^{(2)} - 1 & c_1^{(1)} - 1 \end{vmatrix} \quad \ldots(3.2)$$

provided that $P > N$, $N^{(u)} \geqslant 0$, $u = 1, \ldots, k$.

$$WH[b_1 + 1] - B_1^{(1)} W_1 H[c_1^{(1)} - 1] - \ldots - B_1^{(k)} W_k H[c_1^{(k)} - 1]$$

$$= H \begin{vmatrix} B_1^{(1)} & 0 & \ldots & \ldots & \ldots & 0 & \alpha_1^{(1)} \\ B_1^{(2)} & 0 & \ldots & \ldots & \ldots & \alpha_1^{(2)} & 0 \\ \vdots & \vdots & & & & \vdots & \vdots \\ B_1^{(k)} & \alpha_1^{(k)} & \ldots & \ldots & \ldots & 0 & 0 \\ b_1 & c_1^{(k)} - 1 & \ldots & \ldots & \ldots & c_1^{(2)} - 1 & c_1^{(1)} - 1 \end{vmatrix} \quad \ldots(3.3)$$

provided that $Q \geqslant 0$, $N^{(u)} \geqslant 1$, $u = 1, \ldots, k$.

Derivation of the Recurrence Relations (3.1), (3.2) *and* (3.3)

We prove the recurrence formula (3.1) only as the proofs of (3.2) and (3.3) are obtained in the same way.

First we consider the following recurrence with the undetermined coefficients $L, M_1, \dots, M_k$ and R,

$$LH[a_1 - 1] + M_u H[c_1^{(u)} - 1] = RH.$$

Now we find that by replacing a_1 by $a_1 - 1$ on the right-hand side of (1.1), an extra multiplying factor $1 - a_1 + A_1^{(1)} s_1 + \dots + A_1^{(k)} s_k$ is introduced. Similarly, if we write $c_1^{(u)} - 1$ in place of $c_1^{(u)}$, extra multiplying factors of the type

$$1 - c_1^{(u)} + \alpha_1^{(u)} s_u,\ u = 1, \dots, k$$

are obtained. Therefore, we choose the undetermined coefficients such that

$$L(1 - a_1 + \sum_{u=1}^{k} A_1^{(u)} s_u) + M_u(1 - c_1^{(u)} + \alpha_1^{(u)} s_u) = R,$$

be an indentity in s_u, $u = 1, \dots, k$.

By substituting the values of the undetermined coefficients, the required recurrence formula (3.1) is obtained.

Recurrence formulae (3.2) and (3.3) are derived by proceeding in a similar way.

For this purpose we note that, by putting $a_p - 1$ for a_p on the right-hand side of (1.1) the additional multiplying factor, $a_p - 1 - A_p^{(1)} s_1 - \dots - A_p^{(k)} s_k$, is introduced.

And if we write $b_1 + 1$ for b_1 as before, we obtain

$$- b_1 + B_1^{(1)} s_1 + B_1^{(2)} s_2 + \dots + B_1^{(k)} s_k$$

as the extra multiplying factor. The formula $\Gamma(z + 1) = z\Gamma(z)$ is also used.

By proper combinations of the above mentioned contiguous relations, several other recurrence formulas may be obtained.

As a special case, if we put $N = A = B = 0$, the second member of (1.1) degenerates immediately into k independent, and single, Mellin-Barnes contour integrals each representing the familiar H-function due to Fox (1961).

By suitably adjusting parameters, results of Tandon (1980a) are obtained, which he derived by following a different method due to Srivastava (1966).

4. Particular Cases

If we set $k = 2$, $A_j^{(1)} = \alpha_j$, $A_j^{(2)} = A_j$, $B_j^{(1)} = \beta_j$, $B_j^{(2)} = B_j$

$$\alpha_j^{(1)} = \gamma_j,\ \alpha_j^{(2)} = E_j,\ \beta_j^{(1)} = \delta_j,\ \beta_j^{(2)} = F_j,$$

$$c_j^{(1)} = c_j,\ c_j^{(2)} = e_j,\ d_j^{(1)} = d_j,\ d_j^{(2)} = f_j,$$

$$N = n_1,\ P = p_1,\ Q = q_1$$

and replace $M^{(u)}$ by m_{u+1}, $N^{(u)}$ by n_{u+1}, $P^{(u)}$ by p_{u+1}, $Q^{(u)}$ by q_{u+1}, $(u = 1, 2)$ results of Gupta and Buschman (1975) are obtained.

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