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EXPANSION OF INCOMPLETE GAMMA FUNCTION IN TERMS OF JACOBI POLYNOMIALS

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Incomplete Gamma function is expressed in terms of a rapidly convergent series of Jacobi polynomials. The result is valuable for computations in numerical problems. An integral involving Jacobi polynomials is evaluated for the purpose.

1. INTRODUCTION

Approximations to incomplete Gamma functions are being used in various physical calculations and it is found that the expression of the function in terms of a rapidly convergent series is very valuable for the purpose.

An attempt by Jain (1975) is noteworthy. He has expressed the function in terms of a series of Gegenbauer Polynomials which converges quite rapidly. We attempt the expansion in terms of Jacobi polynomials which give the freedom of one more parameter to the approximation.

The result of Jain (1975) and expansion of incomplete Gamma function in terms of Legendre polynomials follow as particular cases.

2. RESULTS

To obtain the required expansion we establish the following results:

$$\int_{-1}^{1} (1-y)^{\alpha} (1+y)^{\beta} P_n^{(\alpha,\beta)}(y)\, e^{-py} dy$$

$$= (-1)^n 2^{n+\alpha+\beta+1} p^n (n!)^{-1} B(\alpha+n+1, \beta+n+1)\, e^p\, {}_1F_1\left[\begin{matrix}\beta+n+1; \\ \alpha+\beta+2n+2;\end{matrix}\ -2p\right] \quad \ldots(2.1)$$

$$\int_{-1}^{1} (1-y)^{\alpha} (1+y)^{\beta} y^{\lambda-1} e^{-py} P_n^{(\alpha,\beta)}(y)\, dy$$

$$= (-1)^{n+\lambda-1} 2^{\alpha+\beta+1} (n!)^{-1} \Gamma(\alpha+n+1)$$

$$\times e^p (D_p+1)^{\lambda-1} G_{12}^{11}\left(2p \,\middle|\, \begin{matrix} -\beta, \\ n, -\alpha-\beta-n-1 \end{matrix}\right) \quad \ldots(2.2)$$

where $D_p = \frac{d}{dp}$ ·

To prove (2.1) we use Rodrigue's formula [Rainville 1960, p. 257(7)], integration by parts n times and the results of Erdelyi (1954a, 4.3(23) and p. 386].

To establish (2.2) we differentiate (2.1) $(\lambda - 1)$ times w.r.t. p and use the results of Erdelyi (1954b, p. 435).

3. Expansion of Incomplete Gamma Function

Incomplete Gamma function $\gamma(\lambda, x)$ is defined by

$$\gamma(\lambda, x) = \int_0^x e^{-t} t^{\lambda-1}\, dt,\ \lambda > 0. \qquad ...(3.1)$$

Substituting $t = py$ in (3.1), we get

$$\gamma(\lambda, x) = p^{\lambda} \int_0^{x/p} e^{-py}\, y^{\lambda-1}\, dy. \qquad ...(3.2)$$

Let λ be a positive integer and put

$$e^{-py}\, y^{\lambda-1} = \sum_{n=0}^{\infty} A_n P_n^{(\alpha,\beta)}(y) \qquad ...(3.3)$$

where $P_n^{(\alpha,\beta)}(y)$ is Jacobi polynomial of degree precisely n (Rainville 1960, p. 254).

Multiplying both sides of (3.3) by $(1-y)^{\alpha}(1+y)^{\beta} P_n^{(\alpha,\beta)}(y)$ and integrating w.r.t. y over the interval $(-1, 1)$ we get

$$\int_{-1}^{1} e^{-py}(1-y)^{\alpha}(1+y)^{\beta} y^{\lambda-1} P_n^{(\alpha,\beta)}(y)\, dy$$

$$= A_n \int_{-1}^{1} (1-y)^{\alpha}(1+y)^{\beta} [P_n^{(\alpha,\beta)}(y)]^2\, dy. \qquad ...(3.4)$$

Using the results [5, p. 260(11)] and (2.2) we get after some simplification:—

$$A_n = (-1)^{n+\lambda-1} \frac{(\alpha+\beta+2n+1)\,\Gamma(\alpha+\beta+n+1)}{\Gamma(\beta+n+1)} e^p$$

$$\times (D_p+1)^{\lambda-1} G_{2\,2}^{1\,1}\left(2p \,\middle|\, \begin{matrix} -\beta, \\ n, -\alpha-\beta-n-1 \end{matrix}\right) \qquad ...(3.5)$$

From (3.2) and (3.3) we have

$$\gamma(\lambda, x) = p^{\lambda} \int_0^{x/p} \sum_{n=0}^{\infty} A_n P_n^{(\alpha,\beta)}(y)\, dy.$$

To evaluate the integral on the right-hand side we use the result given in Rainville [1960, p. 263 (2)], Therefore,

$$\gamma(\lambda, x) = p^{\lambda} \sum_{n=0}^{\infty} \frac{2A_n}{\alpha+\beta+n} \left[P_n^{(\alpha-1,\beta-1)}\left(\frac{x}{p}\right) - P_n^{(\alpha-1,\beta-1)}(0) \right] \quad \ldots(3.6)$$

Hence

$$\gamma(\lambda, x) = B_0 + \sum_{n=1}^{\infty} B_n P_n^{(\alpha-1,\beta-1)}\left(\frac{x}{p}\right) \quad \ldots(3.7)$$

where

$$B_n = \frac{2p^{\lambda} A_{n-1}}{\alpha + \beta + n - 1} \quad \ldots(3.8)$$

and

$$B_0 = p^{\lambda} \sum_{n=1}^{\infty} A_{n-1} \frac{(-1)^n (\beta)_{n+1}}{2^n (n+1)!} \, {}_2F_1 \left[\begin{matrix} -n-1, -\alpha-n; & -1 \\ \beta; & \end{matrix} \right]. \quad \ldots(3.9)$$

4. Particular Cases

(1) Putting $\alpha = \beta = \nu - \frac{1}{2}$ in (3.5) we get after some simplification

$$A_n = \frac{(2\nu)_n}{(\nu + \frac{1}{2})_n} a_n \quad \ldots(4.1)$$

where a_n is defined by Jain [1975, (1.6)].

Then (3.3) gives us

$$e^{-py} y^{\lambda-1} = \sum_{n=0}^{\infty} a_n C_n^{\nu}(y) \quad \ldots(4.2)$$

which is defined by Jain [1975, (1.3)].

Putting $\alpha = \beta = \nu + \frac{1}{2}$ in (3.7) we get the result obtained by Jain [1975, (1.7)]

(2) Putting $\alpha = \beta = 0$ in (3.5) we get

$$A_n = (-1)^{n+\lambda-1} \frac{n!}{(2n)!} 2^n D_p^{\lambda-1} \left[e^p p^n {}_1F_1 \left(\begin{matrix} 1+n; & -2p \\ 2+2n; & \end{matrix} \right) \right] \quad \ldots(4.3)$$

and (3.7) gives us

$$\gamma(\lambda, x) = B_0 + \sum_{n=1}^{\infty} B_n P_n\left(\frac{x}{p}\right) \qquad \ldots(4.4)$$

where

$$B_n = p^\lambda \left(\frac{A_n - 1}{2n - 1} - \frac{A_n + 1}{2n + 3}\right), n \geqslant 1 \qquad \ldots(4.5)$$

and

$$B_0 = \frac{-A_1}{3} p^\lambda + p^\lambda \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2})_n}{2(n+1)!} A_{2n+1}. \qquad \ldots(4.6)$$

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