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CERTAIN TRANSFORMATIONS OF GENERALISED HYPERGEOMETRIC SERIES

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In this paper we have defined generalisation of the function due to Louck and Biedenharn (1977) to several parameters and derived transformations for that function. Our generalisation can be considered as a formal series in which covergence question played no role. Since the function defined by Louck and Biedenharn (1977) is not arbitrary, having been dictated more or less by the structure found in group theoretic problem (Biedenharn and Louck 1972, Louck *et al.* 1975), so our generalisation is a step forward in the systematic study of the hypergeometric function by this new approach.

1. INTRODUCTION

The hypergeometric series has a long history of physical and mathematical application. Because of the numerous applications and its intrinsic interest to mathematicians this function and its generalisations have been presented in many text-books (Whittaker and Watson 1902, Appell and Kampe de Feriet 1926, Bailey 1935, Erdelyi *et al.* 1953, Rainville 1960, Slater 1968), where reference to the extensive literature on the subject may be found. By recent publications Miller (1968, 1972, 1974) it is evident that the subject is still active.

Recently, Louck and Biedenharn (1977) have defined a new type of hypergeometric function (2.1) with three parameters and t variables. It depends on the properties of the Schur function, which consists of the ratio of two determinants, of the variables $z_1 \ldots z_t$. With this new type of function Louck and Biedenharn (1977) derived generalisation of Saalschutz theorem and proved that the Euler identity holds for this.

In this article we have further generalised hypergeometric function (2.1) due to Louck and Biedenharn (1977) to many parameters and derived certain transformations for this function.

2. GENERALIZATION OF FUNCTION ${}_2\mathcal{F}_1\,(a, b; c; z)$

Louck and Biedenharn (1977) defined the generalised hypergeometric function ${}_2\mathcal{F}_1\,(a, b; c; z)$ by

$$ {}_2\mathcal{F}_1(a, b; c; z) = \sum_{\mu} \langle\, {}_2\mathcal{F}_1(a, b; c)/\mu \,\rangle \langle\, \mu/z \,\rangle, \qquad \ldots(2.1)$$

where the symbol $\langle {}_2\mathcal{F}_1(a, b; c)/\mu \rangle$ depending on the three complex parmeters a, b, c $(c \neq t-1, t-2, \ldots, 0, -1, -2, \ldots)$ and the partition μ in defined as

$$\langle {}_2\mathcal{F}_1(a, b; c)/\mu \rangle = M^{-1}(\mu) \prod_{s=1}^{t} (a-s+1)_{\mu_s}$$

$$\times (b-s+1)_{\mu_s}/(c-s+1)_{\mu_s} \qquad \ldots(2.2)$$

and the factor $M(\mu)$ itself defined by

$$M(\mu) = \prod_{s=1}^{t} (\mu_s + t - s)!/\prod_{r<s} (\mu_r - \mu_s + s - r). \qquad \ldots(2.3)$$

The definition of the Schur function used for defining (2.1) is as follows:

Let $\lambda, \mu, \nu \ldots$ denote partitions of length t that is ordered set of nonnegative integers which satisfy $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_t \geqslant 0$ and let $z = (z_1, \ldots, z_t)$. The Schur function $\langle \lambda/z \rangle$ may be defined by (Littlewood 1950)

$$\langle \lambda/z \rangle = | z_s^{\lambda_k+t-k} | / | z_s^{t-k} | \qquad \ldots(2.4)$$

where $| z_s^{t-k} |$ denotes the Vandermonde determinant

$$| z_s^{t-k} | = \begin{vmatrix} z_1^{t-1}, & z_1^{t-2}, \ldots 1 \\ z_t^{t-1}, & z_t^{t-2}, \ldots 1 \end{vmatrix} \qquad \ldots(2.5)$$

and $| z_s^{\lambda_k+t-k} |$ denotes the determinant

$$| z_s^{\lambda_k+t-k} | = \begin{vmatrix} z_1^{\lambda_1+t-1}, & z_1^{\lambda_2+t-2}, & \ldots, & z_1^{\lambda_t} \\ z_t^{\lambda_1+t-1}, & z_t^{\lambda_2+t-2}, & \ldots, & z_t^{\lambda_t} \end{vmatrix}. \qquad \ldots(2.6)$$

The only property of the Schur function which we require is the multiplication rule (Littlewood 1950) given by

$$\langle \mu/z \rangle \langle \nu/z \rangle = \sum_{\lambda} g(\mu\nu\lambda) \langle \lambda/z \rangle \qquad \ldots(2.7)$$

where $g(\mu\nu\lambda)$ denotes the number of times the irreducible representation λ of the general linear group $GL(t)$ is contained in the direct product representation.

We shall now define (2.1) for many parameters and give its alternative forms.

Let us define

$${}_p\mathcal{F}_q((a_p); (b_q); z) = \sum_{\mu} \langle {}_p\mathcal{F}_q((a_p); (b_q))/\mu \rangle \langle \mu/z \rangle \qquad \ldots(2.8)$$

where the symbol $\langle\, {}_p\mathcal{F}_q((a_p);(b_q))/\mu \,\rangle$ depends on the $p+q$ complex parameters $a_1 \ldots a_p, b_1 \ldots b_q(b_1 \ldots b_q \neq t-1, \ldots 0, -1, -2 \ldots)$ and partition μ :

$$\langle\, {}_p\mathcal{F}_q((a_p);(b_q))/\mu \,\rangle = M^{-1}(\mu) \prod_{s=1}^{t} (a_1 - s + 1)_{\mu_s} \ldots$$

$$\ldots (a_p - s + 1)_{\mu_s}/(b_1 - s + 1)_{\mu_s} \ldots$$

$$\ldots (b_q - s + 1)_{\mu_s} \qquad \ldots(2.9)$$

and $M(\mu)$ is given by (2.3).

But generally it will be convenient to express definition (2.8) in another form involving determinants and hypergeometric function.

Let us introduce the notation

$$A_s = (\mu_s + t - s) \qquad (s = 1, 2, \ldots, t) \qquad \ldots(2.10)$$

$$A = (A_1, A_2, \ldots, A_t) \qquad \ldots(2.11)$$

$$\Delta(z) = \begin{vmatrix} z_1^{t-1}, \ldots, z_1, 1 \\ z_t^{t-1}, \ldots, z_t, 1 \end{vmatrix} = {}_r\mathcal{F}_s(z_r - z_s) \qquad \ldots(2.12)$$

$$\Delta(A; z) = \begin{vmatrix} z_1^{A_1}, z_1^{A_2}, \ldots, z_1^{A_t} \\ z_t^{A_1}, z_t^{A_2}, \ldots, z_t^{A_t} \end{vmatrix} .$$

Multiplying definition (2.8) by

$$\Delta(z) \prod_{s=1}^{t} (a_1 - t + 1)_{t-s} \ldots$$

$$\ldots (a_p - t + 1)_{t-s}/(b_1 - t + 1)_{t-s} \ldots (b_q - t + 1)_{t-s}$$

we easily get

$$\Delta(z)\, {}_p\mathcal{F}_q((a);(b_q); z) \prod_{s=1}^{t} (a_1 - t + 1)_{t-s} \ldots$$

$$\ldots (a_p - t + 1)_{t-s}/(b_1 - t + 1)_{t-s} \ldots (b_q - t + 1)_{t-s}$$

$$= \sum_{A_1 > A_2 > > A_t \geqslant 0} \Delta(A)\, \Delta(A; z) \prod_{s=1}^{t} \frac{(a_1 - t + 1)_{A_s} \ldots (a_p - t + 1)_{A_s}}{A_s!(b_1 - t + 1)_{A_s} \ldots (b_q - t + 1)_{A_s}}$$

$$\ldots(2.13)$$

We now observe what happens to the right-hand side of (2.13) when we extend the summation to all values $\infty \geqslant As \geqslant 0$:

(i) Each term having $Ar = As (r \neq s)$ vanishes in consequence of the factor $\Delta(A)$.

(ii) Each term having indicies $A_{i_1} \ldots A_{i_t}$ where $i_1, \ldots, i_t$ is a permulation of $1, 2, \ldots t$ equals the term having indicies $A_1 > A_2 > \ldots > A_t$.

These two properties imply that we may replace the summation appearing in (2.13) by the summation

$$(1/t\,!) \sum_{A} \equiv (1/t\,!) \sum_{A_1=0}^{\infty} \ldots \sum_{A_t=0}^{\infty} \qquad \ldots(2.14)$$

Further properties of right-hand side of (2.13):

(i) The factor $\prod_{s=1}^{t} (a_1 - t + 1)_{As} \ldots (a_p - t + 1)_{As}/As\,!\,(b_1 - t + 1)_{As} \ldots$

$$\ldots (b_q - t + 1)_{As}$$

may be taken into the determinant $\Delta(A; z)$ where we take sth factor into sth column.

(ii) Using obvious column operations, we may replace $\Delta(A)$ by

$$\Delta(A) = \begin{vmatrix} [A_1]_{t-1} & [A_1]_{t-2} \ldots & [A_1]_0 \\ [A_t]_{t-1} & [A_t]_{t-2} \ldots & [A_t]_0 \end{vmatrix}$$

where $[x]_n = x(x-1) \ldots (x-n+1)$.

Thus we have

$$t\,!\;\Delta(z)\;{}_p\mathcal{F}_q((a_p); (b_q); z) \prod_{s=1}^{t} \frac{(a_1 - t + 1)_{t-s} \ldots (a_p - t + 1)_{t-s}}{(b_1 - t + 1)_{t-s} \ldots (b_q - t + 1)_{t-s}}$$

$$= \sum_{A} \begin{vmatrix} [A_1]_{t-1} \ldots [A_1]_0 \\ [A_t]_{t-1} \ldots [A_t]_0 \end{vmatrix} \begin{vmatrix} f_{A_1}(z_1) \ldots f_{A_t}(z_1) \\ f_{A_1}(z_t) \ldots f_{A_t}(z_t) \end{vmatrix} \qquad \ldots(2.15)$$

where

$$f_k(\xi) = \frac{(a_1 - t + 1)_k \ldots (a_p - t + 1)_k}{(b_1 - t + 1)_k \ldots (b_q - t + 1)_k} \left(\frac{\xi^k}{k!}\right). \qquad \ldots(2.16)$$

Expanding the two determinants occurring in the right-hand side of (2.15) we get R.H.S. equal to

$$= \sum_{A} \sum_{(J_1 \ldots J_t)} \sum_{(i_1 \ldots i_t)} \epsilon^{i_1 \ldots i_t}_{1,2,\ldots,t} \epsilon^{J_1 \ldots J_t}_{1,2,\ldots,t} [A_1]_{t-J_t} \ldots$$

$$\ldots [A_t]_{t-J_t} f_{A_1}(z_{i_1}) \ldots f_{A_t}(z_{i_t})$$

$$= \sum_{(J_1 \ldots J_t)} \sum_{(i_1 \ldots i_t)} \epsilon^{i_1 \ldots i_t}_{1,2,\ldots,t} \epsilon^{J_1 \ldots J_t}_{1,2,\ldots,t} g_{t-J_1}(z_{i_1}) \ldots g_{t-J_t}(z_{i_t}). \qquad \ldots(2.17)$$

$$= t! \begin{vmatrix} g_{t-1}(z_1) \ldots g_0(z_1) \\ g_{t-1}(z_t) \ldots g_0(z_t) \end{vmatrix} \qquad \ldots(2.18)$$

where we have defined

$$g_{t-s}(\xi) = \sum_{k=0}^{\infty} [k]_{t-s} f_k(\xi), \; s = 1, 2, \ldots, t. \qquad \ldots(2.19)$$

Substituting values of $f_k(\xi)$ from (2.16) in (2.19), we get

$$g_{t-s}(\xi) = \frac{(a_1 - t + 1)_{t-s} \ldots (a_p - t + 1)_{t-s}}{(b_1 - t + 1)_{t-s} \ldots (b_q - t + 1)_{t-s}} \xi^{t-s}$$

$$\times \; {}_pF_q\left(\begin{matrix} a_1 - s + 1, \ldots, a_p - s + 1 \\ b_1 - s + 1, \ldots, b_q - s + 1 \end{matrix} ; \xi \right). \qquad \ldots(2.20)$$

Using this result in (2.18) we obtain the following determinantal form for

${}_p\mathscr{F}_q((a_p); (b_q); z)$:

$\Delta(z) \, {}_p\mathscr{F}_q((a_p); (b_q); z)$

$$= \left| z_s^{t-k} \, {}_pF_q\left(\begin{matrix} a_1 - k + 1, \ldots a_p - k + 1 \\ b_1 - k + 1, \ldots b_q - k + 1 \end{matrix} ; z_s \right) \right| \qquad \ldots(2.21)$$

which denotes the element of sth ($s = 1, \ldots, t$) row and kth ($k = 1, 2 \ldots, t$) column of $t \times t$ determinant.

Equation (3.14) of Louck and Biedenharn (1977) is a particular case of (2.21).

3. Transformations of Generalised Series

In this section we shall derive transformation for the function (2.8).

Louck and Biedenharn (1977) proved that the generalised Gauss function obey the Euler identity

$${}_2\mathscr{F}_1(a, b; c; z) \, {}_2\mathscr{F}_1(c - a - b, b; b; z) = {}_2\mathscr{F}_1(c - a, c - b; c; z). \quad \ldots(3.1)$$

Using (3.1) we can easily prove that

$${}_2\mathcal{F}_1(x, y; u; \xi)\, {}_2\mathcal{F}_1(1-n-v, 1-n-w; 1-n-z; \xi)$$
$$= {}_2\mathcal{F}_1(u-x, u-y; u; \xi)\, {}_2\mathcal{F}_1\left(\begin{matrix} v-z, w-z \\ \\ 1-n-z \end{matrix} ; \xi\right) \quad \ldots(3.2)$$

where $u+v+w = x+y+z-n+1$.

Substituting definition (2.1) in (3.2) and using the multiplication rule (2.7) of Schur function, we get

$$\sum_{\lambda,\mu} g(\lambda\mu\nu) \langle {}_2\mathcal{F}_1(x, y; u)/\lambda \rangle \langle {}_2\mathcal{F}_1(1-n-v, 1-n-w;$$
$$1-n-z)/\mu \rangle \langle \nu/z \rangle$$
$$= \sum_{\lambda,\mu} g(\lambda\mu\nu) \langle {}_2\mathcal{F}_1(u-x, u-y; u)/\lambda \rangle \left\langle {}_2\mathcal{F}_1\left(\begin{matrix} v-z, w-z \\ 1-n-z \end{matrix}\right)\Big/\mu \right\rangle \langle \nu/z \rangle$$

i.e. $$\langle {}_2\mathcal{F}_1(x, y; u)/\lambda \rangle \langle {}_2\mathcal{F}_1(1-n-v, 1-n-w; 1-n-z)/\mu \rangle$$
$$= \langle {}_2\mathcal{F}_1(u-x, u-y; u)/\lambda \rangle \langle {}_2\mathcal{F}_1(v-z, w-z; 1-n-z)/\mu \rangle. \quad \ldots(3.3)$$

which is a generalization of the relation connecting terminating Saalschutzian ${}_4F_3$ viz.

$${}_4F_3\left(\begin{matrix} x, y, z, -n; 1 \\ u, v, w \end{matrix}\right) = \frac{(v-z)_n\, (w-z)_n}{(v)_n\, (w_n)}\, {}_4F_3\left(\begin{matrix} u-x, u-y, z, -n; 1 \\ 1-v+z-n,\ 1-w+z-n,\ u \end{matrix}\right).$$

Similarly we can prove

$${}_2\mathcal{F}_1(a, b; c; z)\, {}_2\mathcal{F}_1(1-a, 1-b; 2-c; z)$$
$$= {}_2\mathcal{F}_1(1+a-c, 1+b-c; 2-c; z)\, {}_2\mathcal{F}_1(c-a, c-b; c; z). \quad \ldots(3.4)$$

So the function $\langle {}_2\mathcal{F}_1(a, b, c)/\lambda \rangle$ satisfies the identity

$$\langle {}_2\mathcal{F}_1(a, b; c)/\lambda \rangle \langle {}_2\mathcal{F}_1(1-a, 1-b, 2-c)/\mu \rangle$$
$$= \langle {}_2\mathcal{F}_1(1+a-c, 1+b-c; 2-c)/\lambda \rangle$$
$$\langle {}_2\mathcal{F}_1(c-a, c-b; c)/\mu \rangle. \quad \ldots(3.5)$$

We shall discuss further transformation and generalisations of the known theorems for the function (2.8) in a subsequent communication.

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